EQUIVALENCE OF DOMAINS ARISING FROM DUALITY OF ORBITS ON FLAG MANIFOLDS

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ABSTRACT. S. Gindikin and the author defined a $G_{\mathbb{R}}$-$K_{\mathbb{C}}$ invariant subset $C(S)$ of $G_{\mathbb{C}}$ for each $K_{\mathbb{C}}$-orbit S on every flag manifold $G_{\mathbb{C}}/P$ and conjectured that the connected component $C(S)_0$ of the identity would be equal to the Akhiezer-Gindikin domain D if S is of non-holomorphic type by computing many examples. In this paper, we first prove this conjecture for the open $K_{\mathbb{C}}$-orbit S on an "arbitrary" flag manifold generalizing the result of Barchini. This conjecture for closed S was solved by J. A. Wolf and R. Zierau for Hermitian cases and by G. Fels and A. Huckleberry for non-Hermitian cases. We also deduce an alternative proof of this result for non-Hermitian cases.

1. INTRODUCTION

Let $G_{\mathbb{C}}$ be a connected complex semisimple Lie group and let $G_{\mathbb{R}}$ be a connected real form of $G_{\mathbb{C}}$. Let $K_{\mathbb{C}}$ be the complexification in $G_{\mathbb{C}}$ of a maximal compact subgroup K of $G_{\mathbb{R}}$. Let $X = G_{\mathbb{C}}/P$ be a flag manifold of $G_{\mathbb{C}}$, where P is an arbitrary parabolic subgroup of $G_{\mathbb{C}}$. Then there exists a natural one-to-one correspondence between the set of $K_{\mathbb{C}}$-orbits S and the set of $G_{\mathbb{R}}$-orbits S' on X given by the condition:

$$(1.1) \qquad S \leftrightarrow S' \Longleftrightarrow S \cap S' \text{ is non-empty and compact}$$

([M4]). For each $K_{\mathbb{C}}$-orbit S on X we defined in [GM1] a subset $C(S)$ of $G_{\mathbb{C}}$ by

$$C(S) = \{x \in G_{\mathbb{C}} \mid xS \cap S' \text{ is non-empty and compact}\},$$

where S' is the $G_{\mathbb{R}}$-orbit on X given by (1.1).

Akhiezer and Gindikin defined a domain $D/K_{\mathbb{C}}$ in $G_{\mathbb{C}}/K_{\mathbb{C}}$ in [AG] as follows. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{m}$ denote the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}})$ with respect to K. Let $\mathfrak{t}$ be a maximal abelian subspace in $i\mathfrak{m}$. Put

$$\mathfrak{t}^+ = \{Y \in \mathfrak{t} \mid |\alpha(Y)| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma\},$$

where Σ is the restricted root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}$. Then D is defined by

$$D = G_{\mathbb{R}}(\exp \mathfrak{t}^+)K_{\mathbb{C}}.$$

We conjectured in [GM1] the following.

Conjecture 1.1 (Conjecture 1.6 in [GM1]). *Suppose that $X = G_\mathbb{C}/P$ is not $K_\mathbb{C}$-homogeneous. Then we will have $C(S)_0 = D$ for all $K_\mathbb{C}$-orbits S on X of non-holomorphic type. Here $C(S)_0$ is the connected component of $C(S)$ containing the identity.*

Remark 1.2. (i) When $G_\mathbb{R}$ is of Hermitian type, there exist two special closed $K_\mathbb{C}$-orbits $S_1 = K_\mathbb{C}B/B = Q/B$ and $S_2 = K_\mathbb{C}w_0B/B = w_0Q/B$ on the full flag manifold $G_\mathbb{C}/B$, where $Q = K_\mathbb{C}B$ is the usual maximal parabolic subgroup of $G_\mathbb{C}$ defined by a non-trivial central element in $i\mathfrak{k}$ and w_0 is the longest element in the Weyl group. For each parabolic subgroup P containing B, two closed $K_\mathbb{C}$-orbits S_1P and S_2P on $G_\mathbb{C}/P$ are called of holomorphic type and all the other $K_\mathbb{C}$-orbits are called of non-holomorphic type. When $G_\mathbb{R}$ is of non-Hermitian type, we define that all the $K_\mathbb{C}$-orbits are of non-holomorphic type.

(ii) If $X = G_\mathbb{C}/P$ is $K_\mathbb{C}$-homogeneous, then we have $S = S' = X$ and therefore $C(S) = G_\mathbb{C}$. So we must assume that X is not $K_\mathbb{C}$-homogeneous. When $G_\mathbb{C}$ is simple, it is shown in [O], Theorem 6.1 that there are only two types of $K_\mathbb{C}$-homogeneous flag manifolds X as follows. (Note that the $K_\mathbb{C}$-orbit structure on X depends only on the Lie algebras if $K_\mathbb{C}$ is connected.)

(1) $G_\mathbb{C} = SL(2n, \mathbb{C})$, $K_\mathbb{C} = Sp(n, \mathbb{C})$, $X = P^{2n-1}(\mathbb{C})$.

(2) $G_\mathbb{C} = SO(2n, \mathbb{C})$, $K_\mathbb{C} = SO(2n-1, \mathbb{C})$, $X = SO(2n)/U(n)$.

Let S_{op} denote the unique open $K_\mathbb{C}$-B double coset in $G_\mathbb{C}$. Then S'_{op} is closed in $G_\mathbb{C}$ and therefore we can write

$$C(S_{\mathrm{op}}) = \{x \in G_\mathbb{C} \mid xS_{\mathrm{op}} \supset S'_{\mathrm{op}}\}.$$

The domain $C(S_{\mathrm{op}})_0$ is often called the "Iwasawa domain".

It is proved by Barchini ([B]) that $C(S_{\mathrm{op}})_0 \subset D$. On the other hand, Huckleberry ([H]) proved the opposite inclusion

$$(1.2) \qquad\qquad\qquad\qquad D \subset C(S_{\mathrm{op}})_0.$$

(Recently [M7] gave a proof of (1.2) without complex analysis.) So we have the equality

$$(1.3) \qquad\qquad\qquad\qquad C(S_{\mathrm{op}})_0 = D.$$

It is proved in Proposition 8.1 and Proposition 8.3 in [GM1] that $C(S_{\mathrm{op}})_0 \subset C(S)_0$ for all $K_\mathbb{C}$-orbits S on all flag manifolds $X = G_\mathbb{C}/P$. So we have only to prove the inclusion $C(S)_0 \subset D$ in Conjecture 1.1.

The first aim of this paper is the following generalization of Barchini's theorem which solves Conjecture 1.1 for the open $K_\mathbb{C}$-orbit on an arbitrary flag manifold $G_\mathbb{C}/P$.

Theorem 1.3. *Suppose that $G_\mathbb{R}$ is simple. Then there exists a $K_\mathbb{C}$-B invariant subset $\widetilde{S}$ in $G_\mathbb{C}$ satisfying the following three conditions:*

(i) *$\widetilde{S}$ consists of a single $K_\mathbb{C}$-B double coset when $G_\mathbb{R}$ is of non-Hermitian type and consists of two $K_\mathbb{C}$-B double cosets when $G_\mathbb{R}$ is of Hermitian type.*

(ii) *$x\widetilde{S}^{cl} \cap S'_{\mathrm{op}} \neq \emptyset$ for all elements x in the boundary of D.*

(iii) *Let P be a parabolic subgroup of $G_\mathbb{C}$ containing B. If $G_\mathbb{C}/P$ is not $K_\mathbb{C}$-homogeneous, then*

$$\widetilde{S} \cap S_{\mathrm{op}}P = \emptyset.$$

Corollary 1.4. *Let P be a parabolic subgroup of $G_\mathbb{C}$ containing B. If $G_\mathbb{C}/P$ is not $K_\mathbb{C}$-homogeneous, then $C(S_{\mathrm{op}}P)_0 = D$.*

Proof. Let x be an element in the boundary of D. Then it follows from Theorem 1.3 that $x\widetilde{S}^{cl} \cap S'_{\mathrm{op}} \neq \emptyset$. If $x \in C(S_{\mathrm{op}}P)_0$, then we have

$$xS_{\mathrm{op}}P \supset S'_{\mathrm{op}}P.$$

Hence we have $x\widetilde{S}^{cl} \cap xS_{\mathrm{op}}P \neq \emptyset$. Since $S_{\mathrm{op}}P$ is open in $G_\mathbb{C}$, this implies that $\widetilde{S} \cap S_{\mathrm{op}}P \neq \emptyset$ a contradiction to the condition (iii) in Theorem 1.3. Thus we have $C(S_{\mathrm{op}}P)_0 \subset D$. $\square$

Example 1.5. Let $G_\mathbb{C} = SL(3,\mathbb{C})$, $G_\mathbb{R} = SU(2,1)$ and

$$K_\mathbb{C} = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in G_\mathbb{C} \right\}.$$

Then the complex symmetric space $G_\mathbb{C}/K_\mathbb{C}$ is identified with the space consisting of the pairs (V_+, V_-) of two-dimensional subspaces V_+ and one-dimensional subspaces V_- of $\mathbb{C}^3$ such that $V_+ \cap V_- = \{0\}$ by the identification

$$gK_\mathbb{C} \mapsto (V_+, V_-) = (gV_+^0, gV_-^0).$$

Here $V_+^0 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $V_-^0 = \mathbb{C}e_3$ with the canonical basis e_1, e_2, e_3 of $\mathbb{C}^3$. Using the Hermitian form $Q(z) = |z_1|^2 + |z_2|^2 - |z_3|^2$ defining $SU(2,1)$, we can decompose $\mathbb{C}^3$ as

$$\mathbb{C}^3 = C_0 \sqcup C_+ \sqcup C_-,$$

where $C_0 = \{z \in \mathbb{C}^3 \mid Q(z) = 0\}$, $C_+ = \{z \in \mathbb{C}^3 \mid Q(z) > 0\}$ and $C_- = \{z \in \mathbb{C}^3 \mid Q(z) < 0\}$. Then the Akhiezer-Gindikin domain $D/K_\mathbb{C}$ is described as

$$D/K_\mathbb{C} = \{(V_+, V_-) \in G_\mathbb{C}/K_\mathbb{C} \mid V_+ - \{0\} \subset C_+ \text{ and } V_- - \{0\} \subset C_-\}$$

by [GM1], Proposition 2.2. Hence the boundary of $D/K_\mathbb{C}$ consists of the three $G_\mathbb{R}$-orbits

$$D_1 = \{(V_+, V_-) \in G_\mathbb{C}/K_\mathbb{C} \mid V_+ \text{ is tangent to } C_0 \text{ and } V_- - \{0\} \subset C_-\},$$
$$D_2 = \{(V_+, V_-) \in G_\mathbb{C}/K_\mathbb{C} \mid V_+ - \{0\} \subset C_+ \text{ and } V_- \in C_0\},$$
$$D_3 = \{(V_+, V_-) \in G_\mathbb{C}/K_\mathbb{C} \mid V_+ \text{ is tangent to } C_0 \text{ and } V_- \in C_0\}.$$

Let B denote the standard Borel subgroup of $G_\mathbb{C}$ consisting of upper triangular matrices contained in $G_\mathbb{C}$. Then the full flag manifold $X = G_\mathbb{C}/B$ consists of flags (ℓ, p), where ℓ are one-dimensional subspaces of $\mathbb{C}^3$ and p are two-dimensional subspaces of $\mathbb{C}^3$ containing ℓ. Note that B is the isotropy subgroup of the flag $(\mathbb{C}e_1, \mathbb{C}e_1 \oplus \mathbb{C}e_2)$. We see that X is decomposed into the six $K_\mathbb{C}$-orbits

$$S_1 = \{(\ell, p) \in X \mid \ell = V_-^0\},$$
$$S_2 = \{(\ell, p) \in X \mid p = V_+^0\},$$
$$S_3 = \{(\ell, p) \in X \mid \ell \subset V_+^0 \text{ and } p \supset V_-^0\},$$
$$S_4 = \{(\ell, p) \in X \mid p \supset V_-^0\} - (S_1 \sqcup S_3),$$
$$S_5 = \{(\ell, p) \in X \mid \ell \subset V_+^0\} - (S_2 \sqcup S_3),$$
$$S_{\mathrm{op}} = X - (S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4 \sqcup S_5).$$

On the other hand, the corresponding $G_{\mathbb{R}}$-orbits are

$$S_1' = \{(\ell, p) \in X \mid \ell - \{0\} \subset C_-\},$$
$$S_2' = \{(\ell, p) \in X \mid p - \{0\} \subset C_+\},$$
$$S_3' = \{(\ell, p) \in X \mid \ell - \{0\} \subset C_+ \text{ and } p \cap C_- \neq \emptyset\},$$
$$S_4' = \{(\ell, p) \in X \mid \ell \subset C_0 \text{ and } p \text{ is not tangent to } C_0\},$$
$$S_5' = \{(\ell, p) \in X \mid p \text{ is tangent to } C_0 \text{ and } \ell \not\subset C_0\},$$
$$S_{\mathrm{op}}' = \{(\ell, p) \in X \mid \ell \subset C_0 \text{ and } p \text{ is tangent to } C_0\},$$

respectively.

If $xK_{\mathbb{C}} = (V_+, V_-) \in D_1 \sqcup D_3$, then V_+ is tangent to C_0. Hence the flag $(V_+ \cap C_0, V_+)$ is contained in $xS_2 \cap S_{\mathrm{op}}'$. On the other hand, if $xK_{\mathbb{C}} = (V_+, V_-) \in D_2 \sqcup D_3$, then $V_- \subset C_0$. Hence the flag (V_-, p) (p is tangent to C_0) is contained in $xS_1 \cap S_{\mathrm{op}}'$. Thus we have

$$x(S_1 \sqcup S_2) \cap S_{\mathrm{op}}' \neq \emptyset$$

for all elements x in the boundary of D.

There are two non-trivial parabolic subgroups

$$P_1 = \{g \in G_{\mathbb{C}} \mid gV_+^0 = V_+^0\} \quad \text{and} \quad P_2 = \{g \in G_{\mathbb{C}} \mid g\mathbb{C}e_1 = \mathbb{C}e_1\}$$

in $G_{\mathbb{C}}$ containing B. We see that $S_{\mathrm{op}}P_1 = S_5 \sqcup S_{\mathrm{op}}$ and that $S_{\mathrm{op}}P_2 = S_4 \sqcup S_{\mathrm{op}}$. So we have

$$(S_1 \sqcup S_2) \cap S_{\mathrm{op}}P = \emptyset$$

for all parabolic subgroups P of $G_{\mathbb{C}}$ such that $B \subset P \neq G_{\mathbb{C}}$. Thus we have verified Theorem 1.3 for $\widetilde{S} = S_1 \sqcup S_2$ in this case.

Remark 1.6. (i) The following statement is false:

$$x \in \partial(C(S)_0) \Longrightarrow xS^{cl} \cap \partial S' \neq \emptyset$$

for non-open S. In fact there is a counterexample when $G_{\mathbb{R}} = SU(2, 1)$ as follows. In the above example, let $xK_{\mathbb{C}} = (V_+, V_-)$ be a point in D_1. Then V_+ is tangent to C_0 and $V_- - \{0\} \subset C_-$. Consider the $K_{\mathbb{C}}$-orbit S_4 on $G_{\mathbb{C}}/B$. Then the intersection $xS_4 \cap S_4'$ consists of the flags (ℓ, p) such that

$$\ell \subset C_0, \ \ell \notin V_+ \text{ and } p \supset V_-.$$

Hence $xS_4 \cap S_4'$ is not closed in $G_{\mathbb{C}}$ and therefore $x \in \partial(C(S_4)_0)$. On the other hand, since xS_4^{cl} consists of flags (ℓ, p) satisfying $p \supset V_-$, it does not intersect $\partial S_4' = S_{\mathrm{op}}'$.

(ii) By the above argument we see that

$$(1.4) \qquad\qquad x \in D_1 \sqcup D_3 \Longrightarrow (xS_4 \cap S_4')^{cl} \supset xS_3 \cap S_4'.$$

On the other hand suppose that $x \in D_2 \sqcup D_3$. Then $V_- \subset C_0$. Let (ℓ_0, p_0) denote the unique flag in $xS_1 \cap S_{\mathrm{op}}'$ given by

$$\ell_0 = V_- \quad \text{and} \quad p_0 \text{ is tangent to } C_0.$$

Then we can take a sequence $\{(\ell_n, p_n)\}$ of flags in $xS_4 \cap S_4'$ converging to (ℓ_0, p_0). Hence

$$(1.5) \qquad\qquad x \in D_2 \sqcup D_3 \Longrightarrow (xS_4 \cap S_4')^{cl} \supset xS_1 \cap S_{\mathrm{op}}'.$$

By (1.4) and (1.5), we have

$$C(S_4)_0 \subset D.$$

Since the opposite inclusion is known by (1.3) and [GM1], Proposition 8.3, we have

$$C(S_4)_0 = D.$$

On the other hand, we have $C(S_4 P_1)_0 = C(S_1)$ since $S_4 P_1 = S_1 P_1$ is of holomorphic type ([WZ1]). Hence $C(S_4)_0$ is strictly included in $C(S_4 P_1)_0$ though the contradicting converse inclusion and the equality are asserted in [HN], Proposition 6 and Proposition 10, respectively. (Note that the projection $S_4'/B \to S_4' P_1/P_1$ is proper and that S_4^{cl} is left P_1-invariant.)

Next suppose that S is closed. Then S' is open ([M2]) and so the condition

$$xS \cap S' \text{ is non-empty and compact (in } G_{\mathbb{C}}/P)$$

implies

$$xS \subset S'.$$

Hence the set $C(S)_0$ is the cycle space for S' introduced by Wells and Wolf ([WW]). Let B be a Borel subgroup of $G_{\mathbb{C}}$ contained in P. Let $\{S_j \mid j \in J\}$ denote the set of $K_{\mathbb{C}}$-B double cosets in $G_{\mathbb{C}}$ of codimension one and let $T_j = S_j^{cl}$ denote the closure of S_j. Then we defined in [GM2] a subset $J' = J(S)$ of J for S by

$$J' = \{j \in J \mid S(BwB)^{cl} = T_j \text{ for some } w \in W\}$$

and proved the equality

$$(1.6) \qquad\qquad C(S)_0 = \Omega(J'),$$

where

$$\Omega(J') = \{x \in G_{\mathbb{C}} \mid xT_j \cap S'_{\mathrm{op}} = \emptyset \text{ for all } j \in J'\}_0.$$

(See [GM2], Remark 4, for the related paper [HW].) It is also shown in [GM2] that $\bigcup_{j \in J} T_j$ is the complement of S_{op} in $G_{\mathbb{C}}$. So we have the equalities

$$(1.7) \qquad\qquad D = C(S_{\mathrm{op}})_0 = \Omega(J)$$

by (1.3).

The second aim of this paper is to prove the following theorem.

Theorem 1.7. *Suppose that $G_{\mathbb{R}}$ is of non-Hermitian type. Then the $K_{\mathbb{C}}$-B double coset $\widetilde{S}$ given in Theorem 1.3 is contained in $\bigcap_{j \in J} T_j$.*

As a direct consequence of this theorem, we have:

Corollary 1.8. *If $G_{\mathbb{R}}$ is of non-Hermitian type, then $\Omega(J') = D$ for all nonempty subsets J' of J.*

Proof. Since $D = \Omega(J) \subset \Omega(J')$ by (1.7), we have only to show the inclusion $\Omega(J') \subset D$. Let x be an arbitrary element on the boudary of D. Then

$$xT_j \cap S'_{\mathrm{op}} \supset x\widetilde{S}^{cl} \cap S'_{\mathrm{op}} \neq \emptyset \quad \text{for all } j \in J$$

by Theorem 1.3 and Theorem 1.7. So we have $x \notin \Omega(J')$ for any $J' \neq \emptyset$. Hence $\Omega(J') \subset D$. $\qquad\square$

By (1.6) we have:

Corollary 1.9. *If $G_{\mathbb{R}}$ is of non-Hermitian type and S is a closed $K_{\mathbb{C}}$-orbit on an arbitrary flag manifold $X = G_{\mathbb{C}}/P$ such that $S \neq X$, then $C(S)_0 = D$.*

Here we review the history of the inclusion

$$(1.8) \qquad\qquad\qquad C(S)_0 \subset D$$

for closed $K_{\mathbb{C}}$-orbits S on X. (As we explained, the opposite inclusion is already established.)

It was conjectured in [G] (Problem 6) that the Akhiezer-Gindikin domain D would be the universal domain for all Stein extensions of Riemannian symmetric spaces $G_{\mathbb{R}}/K$ and that it would coincide with the cycle spaces $C(S)_0$ and the Iwasawa domain $C(S_{\mathrm{op}})_0$.

When $G_{\mathbb{R}}$ is of Hermitian type, (1.8) was proved in [WZ1] and [WZ2] for closed $K_{\mathbb{C}}$-orbits of non-holomorphic type (cf. Remark 5 in [GM2]). Remark that $C(S)_0$ is bigger than D if S is of holomorphic type (cf. [WZ1] and [GM1]).

Here we should note the following generality: Let $\pi : X' \to X$ be a $G_{\mathbb{C}}$-equivariant surjection between flag manifolds and let S be a closed $K_{\mathbb{C}}$-orbit on X'. Then

$$xS \subset S' \implies x\pi(S) \subset \pi(S').$$

Hence

$$x \in C(S) \implies x \in C(\pi(S))$$

and so we have $C(S) \subset C(\pi(S))$. This implies that we have only to prove the inclusion (1.8) for minimal flag manifolds with non-trivial $K_{\mathbb{C}}$-orbit structure.

In [GM1], (1.8) was proved for typical minimal flag manifolds X with non-trivial $K_{\mathbb{C}}$-orbit structure for all non-Hermitian classical $G_{\mathbb{R}}$ by case-by-case checkings. (Note that they include the complex cases that $G_{\mathbb{R}} = SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$ or $Sp(n,\mathbb{C})$.) We proposed our Conjecture 1.1 (Conjecture 1.6 in [GM1]) by these many explicit computations.

Recently, Fels and Huckleberry ([FH]) gave a general proof of (1.8) for closed S for all non-Hermitian cases by using a complex analytic notion "Kobayashi hyperbolicity". But we need no complex analysis (except Lemma 2.1) in our proof of Theorem 1.7, Corollary 1.8 and Corollary 1.9 in this paper.

The rest of this paper is consructed as follows. In Section 2 and Section 3, we consider real Lie groups G and associated pairs of symmetric subgroups H and H' of G. If $G = G_{\mathbb{C}}$ and $H = K_{\mathbb{C}}$, then $H' = G_{\mathbb{R}}$. In this general setting we defined a generalization of the Akhiezer-Gindikin domain D in [M7]. In Section 2 we define generic elements in the boundary of D generalizing the results in [FH], Section 4. (Remark: In Section 3 of [FH], they studied the Jordan decomposition and elliptic elements of the double coset decomposition $G_{\mathbb{R}}\backslash G_{\mathbb{C}}/K_{\mathbb{C}}$. But these results were already given in [M5] in a more general setting as we explain in Section 2.)

In Section 3 we construct a parabolic subgroup P_Z such that $H'P_Z$ and $Ha_\alpha^{-1}P_Z$ are closed in G. Here a_α is an element of T contained in the boundary of $\exp \mathfrak{t}^+$. Then we prove the key theorem (Theorem 3.2) which asserts that

$$xHa_\alpha^{-1}P_Z \cap H'P_Z \neq \emptyset$$

for all elements x in the boundary of D under some conditions. (If H' is a group of non-Hermitian type and G is the complexification of H', then the conditions are satisfied. If H' is of Hermitian type, then we must also consider P_{-Z}.)

In sections 4 through 7, we assume that G is a complex semisimple Lie group and H' is a connected real form of G. Hence H is the complexification of a maximal compact subgroup of H'. (If we use the notations in Section 1, then we rewrite as $G = G_{\mathbb{C}}$, $H = K_{\mathbb{C}}$ and $H' = G_{\mathbb{R}}$.) In Section 4 and 5 we prove Theorem 1.3. In

Section 6 we prove Theorem 1.7. Section 7 is an appendix for the orbit structure on the full flag manifolds.

Remark 1.10. Recently Conjecture 1.1 is studied for non-closed and non-open orbits in [M8]. In this paper our conjecture is solved for non-Hermitian cases. So the remaining problem in our conjecture is only for non-closed and non-open orbits when $G_\mathbb{R}$ is of Hermitian type (cf. Remark 1.6).

2. GENERIC ELEMENTS IN THE BOUNDARY OF AKHIEZER-GINDIKIN DOMAIN

First we prepare the following lemma on the "continuity" of the eigenvalues. For a complex $m \times m$ matrix A, define a norm $N(A)$ of A by

$$N(A) = \max\{|a_{ij}| \mid i, j = 1, \ldots, m\}.$$

For $\delta > 0$ define a compact neighborhood $U_\delta(A)$ of A by

$$U_\delta(A) = \{B \mid N(B - A) \leq \delta\}.$$

For a matrix A let $f_A(z) = \det(zI - A)$ denote the eigenpolynomial of A.

Lemma 2.1. (i) *Let A be a complex square matrix with an eigenvalue λ. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$B \in U_\delta(A) \implies \text{ there exists an eigenvalue } \mu \text{ of } B \text{ such that } |\mu - \lambda| < \varepsilon.$$

(ii) *Let $A = \{a_{ij}\}$ be a complex $m \times m$ matrix. Then*

$$|\lambda| \leq mN(A)$$

for all the eigenvalues λ of A.

Proof. (i) Take an η so that $0 < \eta \leq \varepsilon$ and that $f_A(z) \neq 0$ for all z on the circle $C_\eta : |z - \lambda| = \eta$. Then there exists a $\delta > 0$ such that

$$\ell = \min\{|f_B(z)| \mid B \in U_\delta(A), \ z \in C_\eta\} > 0.$$

Put $D_\eta = \{z \in \mathbb{C} \mid |z - \lambda| \leq \eta\}$.

Suppose that there exists a $B \in U_\delta(A)$ such that

$$f_B(z) \neq 0 \text{ for all } z \in D_\eta.$$

Then we will get a contradiction as follows. Put $B(t) = A + t(B - A)$ for $0 \leq t \leq 1$ and define t_0 by

$$t_0 = \inf\{t \in [0, 1] \mid f_{B(t)}(z) \neq 0 \text{ for all } z \in D_\eta\}.$$

Then there is a decreasing sequence $\{t_k\}$ in $[t_0, 1]$ such that $\lim_{k \to \infty} t_k = t_0$ and that $g_k(z) = 1/f_{B(t_k)}(z)$ are holomorphic functions on D_η. We have

$$|g_k(z)| \leq \frac{1}{\ell}$$

for $z \in D_\eta$ by the maximum principle. So we have $|f_{B(t_k)}(z)| \geq \ell$ for $z \in D_\eta$ and therefore $|f_{B(t_0)}(z)| \geq \ell$ for $z \in D_\eta$. Moreover if $t_0 > 0$, then there exists a $t \in (0, t_0)$ such that $f_{B(t)}(z) \neq 0$ for $z \in D_\eta$, contradicting the definition of t_0. So we have $t_0 = 0$ and $|f_A(z)| \geq \ell$ for $z \in D_\eta$. But this contradicts the assumption that $f_A(\lambda) = 0$.

(ii) If $|z| > mN(A)$, then we can define

$$(zI - A)^{-1} = \frac{1}{z}\left(I - \frac{1}{z}A\right)^{-1} = \frac{1}{z}\left(I + \frac{1}{z}A + \frac{1}{z^2}A^2 + \cdots\right)$$

since the right-hand side converges. So the eigenvalues λ of A satisfy

$$|\lambda| \leq mN(A).$$

$\square$

Let G be a connected real semisimple Lie group. Let σ be an involution of G (automorphism of G of order two). Then there exists a Cartan involution θ of G such that $\sigma\theta = \theta\sigma$. Denote the corresponding involutions of $\mathfrak{g} = \mathrm{Lie}(G)$ by the same letters as usual. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{and} \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

denote the $+1$ and -1 eigenspace decomposition of $\mathfrak{g}$ for the involutions σ and θ, respectively. Put $\tau = \sigma\theta$. Then τ is also an involution of G. Let

$$\mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{q}'$$

be the $+1$ and -1 eigenspace decomposition of $\mathfrak{g}$ for τ. Then

$$\mathfrak{h}' = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{m} \cap \mathfrak{q}) \quad \text{and} \quad \mathfrak{q}' = (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{m} \cap \mathfrak{h}).$$

Define two symmetric subgroups H and H' as the connected components of G^σ and G^τ, respectively, containing the identity. Then H and H' are called "associated" ([Be], [M1]).

The double coset decomposition $H\backslash G/L$, where H and L are arbitrary two symmetric subgroups of G, is studied in [M5]. Of course we can apply it to the pair of the symmetric subgroups (H, H').

Remark 2.2. $K_\mathbb{C}$ and $G_\mathbb{R}$ are associated in the complex Lie group $G_\mathbb{C}$. So we can consider the setting in Section 1 as a special one.

We defined a generalization of the Akhiezer-Gindikin domain in [M7] as follows. Let $\mathfrak{t}$ be a maximal abelian subspace of $\mathfrak{k} \cap \mathfrak{q}$. Then we can define the root space

$$\mathfrak{g}_\mathbb{C}(\mathfrak{t}, \alpha) = \{X \in \mathfrak{g}_\mathbb{C} \mid [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{t}\}$$

for any linear form $\alpha : \mathfrak{t} \to i\mathbb{R}$. Here $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$ is the complexification of $\mathfrak{g}$. Put

$$\Sigma = \Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{t}) = \{\alpha \in i\mathfrak{t}^* - \{0\} \mid \mathfrak{g}_\mathbb{C}(\mathfrak{t}, \alpha) \neq \{0\}\}.$$

Then Σ satisfies the axiom of the root system ([R], Theorem 5). Since $\theta(Y) = Y$ for all $Y \in \mathfrak{t}$, we can decompose $\mathfrak{g}_\mathbb{C}(\mathfrak{t}, \alpha)$ into the $+1, -1$-eigenspaces for θ as

(2.1) $$\mathfrak{g}_\mathbb{C}(\mathfrak{t}, \alpha) = \mathfrak{k}_\mathbb{C}(\mathfrak{t}, \alpha) \oplus \mathfrak{m}_\mathbb{C}(\mathfrak{t}, \alpha).$$

Define a subset

(2.2) $$\Sigma(\mathfrak{m}_\mathbb{C}, \mathfrak{t}) = \{\alpha \in i\mathfrak{t}^* - \{0\} \mid \mathfrak{m}_\mathbb{C}(\mathfrak{t}, \alpha) \neq \{0\}\}$$

of Σ and put

$$\mathfrak{t}^+ = \{Y \in \mathfrak{t} \mid |\alpha(Y)| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma(\mathfrak{m}_\mathbb{C}, \mathfrak{t})\}.$$

Then we define a generalization of the Akhiezer-Gindikin domain D in G by

$$D = H'T^+H,$$

where $T^+ = \exp \mathfrak{t}^+$. (We showed in [M7], Proposition 1, that D is open in G.) The following assertion is already proved in the proof of [M7], Proposition 2.

Lemma 2.3. $D \cap T = T^+(T \cap H)$.

In [M5] we considered the automorphism

$$f_x = \tau \mathrm{Ad}(x)\sigma \mathrm{Ad}(x)^{-1}$$

of $\mathfrak{g}$ for every element x in G. Then we defined the "Jordan decomposition" $x = (\exp X_n)x_s$ of the element x so that $f_x = f_{x_s}\mathrm{Ad}(\exp(-2X_n)) = \mathrm{Ad}(\exp(-2X_n))f_{x_s}$ is the usual multiplicative Jordan decomposition of the automorphism f_x of $\mathfrak{g}$ and that X_n is a nilpotent element in $\mathfrak{q}' \cap \mathrm{Ad}(x_s)\mathfrak{q}$. It is shown in [M5], Proposition 3, that

$$(2.3) \qquad\qquad H'x_s H \subset (H'xH)^{cl}.$$

Let ∂D denote the boundary of D in G and let ∂T^+ denote the boundary of T^+ in T.

Lemma 2.4. *If* $x \in \partial D$, *then* $H'x_s H = H'aH$ *for some* $a \in \partial T^+$.

Proof. If $y = h'th \in D$ with $h' \in H'$, $t \in T^+$ and $h \in H$, then

$$\begin{aligned} f_y &= \tau \mathrm{Ad}(h'th)\sigma \mathrm{Ad}(h'th)^{-1} \\ &= \mathrm{Ad}(h')\tau\sigma \mathrm{Ad}(t)^{-2}\mathrm{Ad}(h')^{-1} = \mathrm{Ad}(h')\theta \mathrm{Ad}(t)^{-2}\mathrm{Ad}(h')^{-1}. \end{aligned}$$

Hence the absolute values of the eigenvalues of f_y are all equal to one.

Let x be an element of ∂D. By the continuity of the eigenvalues shown in Lemma 2.1(i), the absolute values of the eigenvalues of f_x are all one. This also holds for f_{x_s}. So if we decompose $x_s = (\exp X_p)x_k$ (polar decomposition) as in [M5], Section 4.2, then we have $X_p = 0$ and $x_k = x_s$. Let us call such an element x_s "elliptic" following the terminology in [FH]. We can show that $y \in G$ is elliptic if and only if y is contained in $H'TH$ by the arguments in the proof of [M5], Theorem 2. Write $x_s = h'ah$ with some $h' \in H'$, $a \in T$ and $h \in H$. By (2.3) x_s is contained in D^{cl}. Hence a is also contained in D^{cl}. We have only to show $a \in (D \cap T)^{cl}$ in view of Lemma 2.3.

Since $f_a = \tau \mathrm{Ad}(a)\sigma \mathrm{Ad}(a)^{-1}$ is semisimple, we can decompose $\mathfrak{g}$ as

$$(2.4) \qquad\qquad \mathfrak{g} = (\mathfrak{h}' + \mathrm{Ad}(a)\mathfrak{h}) \oplus (\mathfrak{q}' \cap \mathrm{Ad}(a)\mathfrak{q})$$

by [M5], Lemma 1. Let $B(\,,\,)$ denote the Killing form on $\mathfrak{g}$ and let $B_\theta(\,,\,)$ denote the positive definite bilinear form on $\mathfrak{g}$ defined by $B_\theta(X,Y) = -B(X,\theta Y)$. Since $a \in D^{cl}$, by (2.4) we can take an element Y of $\mathfrak{q}' \cap \mathrm{Ad}(a)\mathfrak{q}$ such that $y = (\exp Y)a \in D$ and that $B_\theta(Y,Y) < \varepsilon^2$ for an arbitrary positive real number ε. Since

$$f_y = \tau \mathrm{Ad}(y)\sigma \mathrm{Ad}(y)^{-1} = f_a \mathrm{Ad}(\exp 2Y)^{-1} = \mathrm{Ad}(\exp 2Y)^{-1}f_a$$

is elliptic, it follows that $\mathrm{Ad}(\exp 2Y)$ is elliptic. Taking ε sufficiently small, we may assume that Y is semisimple and that the eigenvalues of $\mathrm{ad}(Y)$ are pure imaginary. Hence there exists an element $h \in H' \cap \mathrm{Ad}(a)H$ such that $Y' = \mathrm{Ad}(h)Y$ is contained in $\mathfrak{t}$ which is a compact Cartan subset of $\mathfrak{q}' \cap \mathrm{Ad}(a)\mathfrak{q}$ (cf. [OM], Corollary of Theorem 2). Since $(\exp Y')a \in D \cap T$ and since $B_\theta(Y',Y') \leq B_\theta(Y,Y) < \varepsilon^2$ by the following lemma, we have proved $a \in (D \cap T)^{cl}$. $\qquad\square$

Lemma 2.5. *If* $Y = \mathrm{Ad}(g)Y'$ *for some* $g \in G$ *and* $Y' \in \mathfrak{k}$, *then* $B_\theta(Y',Y') \leq B_\theta(Y,Y)$.

Proof. Since $B_\theta(\mathrm{Ad}(k)Y', \mathrm{Ad}(k)Y') = B_\theta(Y', Y')$ for $k \in K$, we may assume $g = \exp Z$ for some $Z \in \mathfrak{m}$. Write $Y' = \sum_\lambda Y_\lambda$ with λ-eigenvectors Y_λ for $\mathrm{ad}(Z)$ ($\lambda \in \mathbb{R}$). Since $Y' \in \mathfrak{k}$, we have $\theta Y_\lambda = Y_{-\lambda}$. Hence

$$
\begin{aligned}
B_\theta(Y, Y) &= B_\theta(\mathrm{Ad}(\exp Z)Y', \mathrm{Ad}(\exp Z)Y') = \sum_\lambda e^{2\lambda} B_\theta(Y_\lambda, Y_\lambda) \\
&= B_\theta(Y_0, Y_0) + \sum_{\lambda > 0} (e^{2\lambda} + e^{-2\lambda}) B_\theta(Y_\lambda, Y_\lambda) \\
&\geq B_\theta(Y_0, Y_0) + \sum_{\lambda > 0} 2 B_\theta(Y_\lambda, Y_\lambda) = B_\theta(Y', Y'). \qquad \square
\end{aligned}
$$

Lemma 2.6. *Every $x \in \partial D$ is contained in the boundary of the complement $G - D^{cl}$ of D^{cl} in G.*

Proof. First assume that x is semisimple. Then we may assume $x = a = \exp Y$ with $Y \in \partial \mathfrak{t}^+$ by Lemma 2.4. There exists an $\alpha \in \Sigma$ such that $\alpha(Y) = \pi i/2$ and that $\mathfrak{m}_\mathbb{C}(\mathfrak{t}, \alpha) \neq \{0\}$ by the definition of $\mathfrak{t}^+$. Since $X \mapsto \tau \overline{X}$ defines a conjugation of $\mathfrak{m}_\mathbb{C}(\mathfrak{t}, \alpha)$, we can take a non-zero element X of $\mathfrak{m}_\mathbb{C}(\mathfrak{t}, \alpha)$ such that $\tau \overline{X} = -X$. Then $Z = X + \overline{X}$ is a non-zero element of $\mathfrak{g}^{f_a} \cap \mathfrak{m} \cap \mathfrak{q}'$. Since

$$
\begin{aligned}
f_{(\exp tZ)a} &= \tau \mathrm{Ad}(\exp tZ) \mathrm{Ad}(a) \sigma \mathrm{Ad}(a)^{-1} \mathrm{Ad}(\exp tZ)^{-1} \\
&= \mathrm{Ad}(\exp tZ)^{-1} \tau \mathrm{Ad}(a) \sigma \mathrm{Ad}(a)^{-1} \mathrm{Ad}(\exp tZ)^{-1} \\
&= \mathrm{Ad}(\exp tZ)^{-1} f_a \mathrm{Ad}(\exp tZ)^{-1} = f_a \mathrm{Ad}(\exp tZ)^{-2} = \mathrm{Ad}(\exp tZ)^{-2} f_a
\end{aligned}
$$

and since $\mathrm{Ad}(\exp tZ)^{-2}$ has a non-trivial real eigenvalue for $t \in \mathbb{R}^\times$, it follows that $(\exp tZ)a$ is a non-elliptic semisimple element and hence $(\exp tZ)a \notin D^{cl}$ for $t \in \mathbb{R}^\times$. Thus $a \in \partial(G - D^{cl})$.

Next assume that x is not semisimple. Then we may assume that $x = (\exp X_n)a$ with an $a \in \partial T^+$ and a nilpotent element $X_n \neq 0$ in $\mathfrak{q}' \cap \mathrm{Ad}(a)\mathfrak{q}$ by Lemma 2.4. By a generalization of the Jacobson-Morozov theorem, there exist a $Y \in \mathfrak{h}' \cap \mathrm{Ad}(a)\mathfrak{h}$ and an $X'_n \in \mathfrak{q}' \cap \mathrm{Ad}(a)\mathfrak{q}$ such that

$$
[Y, X_n] = 2X_n, \quad [Y, X'_n] = -2X'_n \quad \text{and} \quad [X_n, X'_n] = Y
$$

([KR], Proposition 4; cf. also [Se], Section 1). For $t \in \mathbb{R}^\times$ we have

$$
[X_n + t^2 X'_n, Y - \frac{1}{t} X_n + t X'_n] = 2t(Y - \frac{1}{t} X_n + t X'_n).
$$

Hence $\mathrm{Ad}(\exp(X_n + t^2 X'_n))$ is semisimple and it has an eigenvalue e^{2t}. From the same argument as in the first case it follows that $(\exp(X_n + t^2 X'_n))a$ is a non-elliptic semisimple element for $t \in \mathbb{R}^\times$. Hence $x = (\exp X_n)a \in \partial(G - D^{cl})$. $\square$

Let $\Sigma(\mathfrak{m}_\mathbb{C}, \mathfrak{t})$ be as in (2.2) and similarly define another subset $\Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{t})$ of Σ by

$$
\Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{t}) = \{\alpha \in i\mathfrak{t}^* - \{0\} \mid \mathfrak{k}_\mathbb{C}(\mathfrak{t}, \alpha) \neq \{0\}\}.
$$

For $\alpha \in \Sigma$ and $k \in \mathbb{R}$ let $p_{\alpha, k}$ denote the hyperplane in $\mathfrak{t}$ defined by

$$
p_{\alpha, k} = \{Y \in \mathfrak{t} \mid \alpha(Y) = k\pi i\}.
$$

Define families $\mathcal{H}_+$ and $\mathcal{H}_-$ of hyperplanes by

$$
\mathcal{H}_+ = \{p_{\alpha, k} \mid \alpha \in \Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{t}), \ k \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{H}_- = \{p_{\alpha, k} \mid \alpha \in \Sigma(\mathfrak{m}_\mathbb{C}, \mathfrak{t}), \ k \in \frac{1}{2} + \mathbb{Z}\},
$$

respectively. Put $\mathcal{H} = \mathcal{H}_+ \sqcup \mathcal{H}_-$. Then the family $\mathcal{H}$ of hyperplanes defines a family $\mathcal{E}$ of Euclidean cells consisting of cells Δ which are maximal connected subsets of $\mathfrak{t}$ satisfying the condition

$$\Delta \subset p \text{ or } \Delta \cap p = \emptyset \text{ for all } p \in \mathcal{H}.$$

We have the cellular decomposition $\mathfrak{t} = \bigsqcup_{\Delta \in \mathcal{E}} \Delta$. Since $\mathfrak{t}^+$ is bounded ([M7], Lemma 1), there exists a finite subset $\mathcal{B}$ of $\mathcal{E}$ such that $\partial \mathfrak{t}^+ = \bigsqcup_{\Delta \in \mathcal{B}} \Delta$. For $\Delta \in \mathcal{E}$ let $\Sigma_\Delta^\pm$ denote the subset of Σ given by

$$\Sigma_\Delta^\pm = \{ \alpha \in \Sigma \mid \Delta \subset p_{\alpha,k} \text{ for some } p_{\alpha,k} \in \mathcal{H}_\pm \}$$

and put $\Sigma_\Delta = \Sigma_\Delta^+ \sqcup \Sigma_\Delta^-$.

Let Y be an element of a cell Δ and put $a = \exp Y$. We can prove the following lemma by the same argument as in [M7], Lemma 2.

Lemma 2.7. $\mathfrak{g}_{\mathbb{C}}^{f_a} = \mathfrak{z}_{\mathfrak{k}_{\mathbb{C}}}(\mathfrak{t}) \oplus \bigoplus_{\alpha \in \Sigma_\Delta^+} \mathfrak{k}_{\mathbb{C}}(\mathfrak{t}, \alpha) \oplus \bigoplus_{\alpha \in \Sigma_\Delta^-} \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha).$

Proof. By (2.1) we have the decomposition

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in \Sigma \sqcup \{0\}} (\mathfrak{k}_{\mathbb{C}}(\mathfrak{t}, \alpha) \oplus \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha)).$$

If $X \in \mathfrak{k}_{\mathbb{C}}(\mathfrak{t}, \alpha)$, then $f_a(X) = \mathrm{Ad}(a)^{-2}\theta(X) = e^{-2\alpha(Y)}X$. If $X \in \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha)$, then $f_a(X) = \mathrm{Ad}(a)^{-2}\theta(X) = -e^{-2\alpha(Y)}X$. So the assertion is clear. $\square$

This lemma implies that

$$\mathfrak{l}_\Delta = \mathfrak{g}^{f_a} = (\mathfrak{h}' \cap \mathrm{Ad}(a)\mathfrak{h}) \oplus (\mathfrak{q}' \cap \mathrm{Ad}(a)\mathfrak{q})$$

is independent of the choice of Y in Δ. Let $\mathfrak{z}_\Delta$ denote the center of $\mathfrak{l}_\Delta$ and let $\mathfrak{s}_\Delta = [\mathfrak{l}_\Delta, \mathfrak{l}_\Delta]$ denote the semisimple part of $\mathfrak{l}_\Delta$. Then we have $\mathfrak{l}_\Delta = \mathfrak{z}_\Delta \oplus \mathfrak{s}_\Delta$ and therefore

$$(2.5) \qquad \mathfrak{q}' \cap \mathrm{Ad}(a)\mathfrak{q} = \mathfrak{l}_\Delta \cap \mathfrak{q}' = (\mathfrak{z}_\Delta \cap \mathfrak{q}') \oplus (\mathfrak{s}_\Delta \cap \mathfrak{q}').$$

By Lemma 2.7 we have

$$\mathfrak{z}_\Delta \cap \mathfrak{q}' \subset \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}) \cap \mathfrak{q}' = \mathfrak{t}.$$

Hence we can write

$$\mathfrak{z}_\Delta \cap \mathfrak{q}' = \{ Y \in \mathfrak{t} \mid \alpha(Y) = 0 \text{ for all } \alpha \in \Sigma_\Delta \}.$$

This implies that

$$(2.6) \qquad \mathfrak{z}_\Delta \cap \mathfrak{q}' \text{ is the tangent space of } \Delta.$$

We see that the nilpotent variety in $\mathfrak{s}_\Delta \cap \mathfrak{q}'$ is decomposed into a finite number of $(S_\Delta \cap H')_0$-orbits, where S_Δ denotes the analytic subgroup of G for $\mathfrak{s}_\Delta$. (It is shown in [KR], Theorem 2, that there are a finite number of nilpotent $((S_\Delta \cap H')_0)_{\mathbb{C}}$-orbits $\mathcal{N}_{\mathbb{C}}$ in $(\mathfrak{s}_\Delta \cap \mathfrak{q}')_{\mathbb{C}}$. For each $\mathcal{N}_{\mathbb{C}}$ it follows from Whitney's theorem for real algebraic varieties ([Wh], Theorem 3.3 in [PR]) that $\mathcal{N}_{\mathbb{C}} \cap (\mathfrak{s}_\Delta \cap \mathfrak{q}')$ consists of a finite number of connected components which are $(S_\Delta \cap H')_0$-orbits.)

For $\Delta \in \mathcal{B}$ let $\mathcal{N}_d$ denote the union of the nilpotent $(S_\Delta \cap H')_0$-orbits of codimension d which are contained in the closure of the set $(\mathfrak{s}_\Delta \cap \mathfrak{q}')_{ell}$ of the elliptic elements in $\mathfrak{s}_\Delta \cap \mathfrak{q}'$. Define a subset

$$M(\Delta, d) = H'(\exp \mathcal{N}_d)(\exp \Delta)H$$

of G.

Proposition 2.8. (i) $\partial D = \bigsqcup_d \bigcup_{\Delta \in \mathcal{B}} M(\Delta, d)$.

(ii) *If* $\mathcal{N}_d \neq \emptyset$, *then* $M(\Delta, d)$ *is a locally closed d-codimensional submanifold of* G *consisting of a finite number of connected components.*

(iii) $\partial D = (\bigcup_{\Delta \in \mathcal{B}} M(\Delta, 1))^{cl}$.

Proof. (i) Let x be an element of ∂D. By Lemma 2.4 x is contained in

$$H'(\exp X_n)aH$$

for some $a \in \partial T^+$ and a nilpotent element X_n in $\mathfrak{s}_\Delta \cap \mathfrak{q}'$. Here we write $a = \exp Y$ with $Y \in \Delta \subset \partial \mathfrak{t}^+$. We will show that X_n is contained in the closure of $(\mathfrak{s}_\Delta \cap \mathfrak{q}')_{ell}$. Since $S_\Delta \cap H' \subset H' \cap aHa^{-1}$, we can take an $S_\Delta \cap H'$-conjugate of X_n so that X_n is sufficiently close to the origin. If X_n is not on the boundary of $(\mathfrak{s}_\Delta \cap \mathfrak{q}')_{ell}$, then there exists a neighborhood U of X_n in $\mathfrak{s}_\Delta \cap \mathfrak{q}'$ consisting of non-elliptic elements. Take a neighborhood V of 0 in $\mathfrak{z}_\Delta \cap \mathfrak{q}'$. Then it follows from (2.4) and (2.5) that the set

$$H'(\exp U)(\exp V)aH$$

contains a neighborhood of $y = (\exp X_n)a$ in G consisting of non-elliptic elements. Hence y is not contained in the closure of D, contradicting $x \in \partial D$. Thus we have proved that X_n is on the boundary of $(\mathfrak{s}_\Delta \cap \mathfrak{q}')_{ell}$. Conversely, if X_n is on the boundary of $(\mathfrak{s}_\Delta \cap \mathfrak{q}')_{ell}$, then it is clear that $y = (\exp X_n)a \in \partial D$.

(ii) It follows from (2.4), (2.5) and (2.6) that the codimension of $M(\Delta, d)$ in G equals the codimension d of $\mathcal{N}_d$ in $\mathfrak{s}_\Delta \cap \mathfrak{q}'$. Considering the left H'-action and the right H-action, we have only to show that $M(\Delta, d)$ is locally closed at $y = \exp X_n \exp Y_0$ for every $Y_0 \in \Delta$ and every $X_n \in \mathcal{N}_d$. Furthermore taking an $S_\Delta \cap H'$-conjugate, we may assume that X_n is sufficiently close to 0. Let V be a compact neighborhood of Y_0 in Δ. Then the non-trivial eigenvalues of $f_{\exp Y}$ for $Y \in V$ are contained in a compact subset Λ of $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ such that $1 \notin \Lambda$. By Lemma 2.1(ii) we can take a neighborhood U of 0 in $\mathfrak{s}_\Delta \cap \mathfrak{q}'$ such that the eigenvalues of $\mathrm{Ad}\exp(-2Z)$ are not contained in Λ^{-1} for all $Z \in U$. By (2.4) and (2.5) we have only to show that

$$(\exp U \exp V) \cap M(\Delta, d) = \exp(U \cap \mathcal{N}_d)\exp V.$$

Suppose $Z \in U$, $Y \in V$ and $\exp Z \exp Y \in M(\Delta, d)$. Then we have only to show $Z \in \mathcal{N}_d$.

Let $\mathfrak{g}_\mathbb{C}^\lambda$ denote the λ-eigenspace for $f_{\exp Y}$. Then $\mathfrak{g}_\mathbb{C}$ is decomposed as

$$\mathfrak{g}_\mathbb{C} = \bigoplus_\lambda \mathfrak{g}_\mathbb{C}^\lambda$$

since $f_{\exp Y}$ is semisimple. Since

$$f_{\exp Z \exp Y} = f_{\exp Y}\mathrm{Ad}\exp(-2Z) = \mathrm{Ad}\exp(-2Z)f_{\exp Y},$$

we can furthurmore decompose the spaces $\mathfrak{g}_\mathbb{C}^\lambda$ into the generalized eigenspaces as

$$\mathfrak{g}_\mathbb{C}^\lambda = \bigoplus_\mu \mathfrak{g}_\mathbb{C}^{\lambda,\mu},$$

where $\mathfrak{g}_\mathbb{C}^{\lambda,\mu} = \{X \in \mathfrak{g}_\mathbb{C}^\lambda \mid (\mathrm{Ad}\exp(-2Z) - \mu I)^k X = 0 \text{ for some } k\}$. Note that every $X \in \mathfrak{g}_\mathbb{C}^{\lambda,\mu}$ is a generalized eigenvector for $f_{\exp Z \exp Y}$ with the eigenvalue $\lambda\mu$. Since we assume that $\mu \notin \Lambda^{-1}$, we have

$$\lambda\mu = 1 \Longrightarrow \lambda = \mu = 1.$$

This implies that the generalized eigenspace $\mathfrak{g}_{\mathbb{C}}^{(1)}$ of $f_{\exp Z \exp Y}$ for the eigenvalue 1 is contained in $\mathfrak{g}_{\mathbb{C}}^1 = (\mathfrak{l}_\Delta)_{\mathbb{C}}$. On the other hand, the condition $\exp Z \exp Y \in M(\Delta, d) = H'(\exp \mathcal{N}_d)(\exp \Delta)H$ implies that

(2.7) $\qquad\qquad\qquad f_{\exp Z \exp Y}$ is conjugate to $f_{(\exp W)b}$

for some $W \in \mathcal{N}_d$ and $b \in \exp \Delta$. So we have

$$\dim \mathfrak{g}_{\mathbb{C}}^{(1)} = \dim \mathfrak{g}_{\mathbb{C}}^{f_b} = \dim(\mathfrak{l}_\Delta)_{\mathbb{C}}.$$

Hence we have

$$\mathfrak{g}_{\mathbb{C}}^{(1)} = (\mathfrak{l}_\Delta)_{\mathbb{C}}.$$

This implies that the linear map $\mathrm{Ad} \exp(-2Z) : (\mathfrak{l}_\Delta)_{\mathbb{C}} \to (\mathfrak{l}_\Delta)_{\mathbb{C}}$ has the unique eigenvalue 1. Hence $\mathrm{Ad} \exp(-2Z)$ is unipotent on $(\mathfrak{l}_\Delta)_{\mathbb{C}}$ and therefore $Z \in \mathfrak{s}_\Delta \cap \mathfrak{q}'$ is nilpotent.

It follows from (2.7) that

$$\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{f_{\exp Z \exp Y}} = \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{f_{(\exp W)b}}.$$

Since $\mathfrak{g}_{\mathbb{C}}^{f_{\exp Z \exp Y}}$ is contained in $\mathfrak{g}_{\mathbb{C}}^{(1)} = \mathfrak{g}_{\mathbb{C}}^1 = (\mathfrak{l}_\Delta)_{\mathbb{C}}$, we have

$$\mathfrak{g}_{\mathbb{C}}^{f_{\exp Z \exp Y}} = (\mathfrak{l}_\Delta)_{\mathbb{C}}^{\exp Z} = (\mathfrak{l}_\Delta)_{\mathbb{C}}^{Z} = \{X \in \mathfrak{l}_{\mathbb{C}} \mid [X, Z] = 0\}.$$

Similary we also have

$$\mathfrak{g}_{\mathbb{C}}^{f_{(\exp W)b}} = (\mathfrak{l}_\Delta)_{\mathbb{C}}^{W} = \{X \in \mathfrak{l}_{\mathbb{C}} \mid [X, W] = 0\}.$$

Hence we have $\dim_{\mathbb{C}}(\mathfrak{l}_\Delta)_{\mathbb{C}}^{Z} = \dim_{\mathbb{C}}(\mathfrak{l}_\Delta)_{\mathbb{C}}^{W}$ and therefore $\dim_{\mathbb{C}}(\mathfrak{s}_\Delta)_{\mathbb{C}}^{Z} = \dim_{\mathbb{C}}(\mathfrak{s}_\Delta)_{\mathbb{C}}^{W}$. This implies that

$$\dim_{\mathbb{C}} \mathrm{Ad}(S_\Delta)_{\mathbb{C}}Z = \dim_{\mathbb{C}} \mathrm{Ad}(S_\Delta)_{\mathbb{C}}W.$$

By [KR], Proposition 5, we have

$$\dim_{\mathbb{C}} \mathrm{Ad}(S_\Delta \cap H')_{\mathbb{C}}Z = \frac{1}{2} \dim_{\mathbb{C}} \mathrm{Ad}(S_\Delta)_{\mathbb{C}}Z = \frac{1}{2} \dim_{\mathbb{C}} \mathrm{Ad}(S_\Delta)_{\mathbb{C}}W$$
$$= \dim_{\mathbb{C}} \mathrm{Ad}(S_\Delta \cap H')_{\mathbb{C}}W.$$

So we have

(2.8) $\qquad\qquad\qquad \dim \mathrm{Ad}(S_\Delta \cap H')Z = \dim \mathrm{Ad}(S_\Delta \cap H')W.$

Since $W \in \mathcal{N}_d$, it is contained in the closure of $(\mathfrak{s}_\Delta \cap \mathfrak{q}')_{ell}$. Hence $\exp Z \exp Y \in H'(\exp W)aH$ is contained in ∂D. By the argument in the proof of (i), we see that Z is contained in the closure of $(\mathfrak{s}_\Delta \cap \mathfrak{q}')_{ell}$. Combining with (2.8), we have proved $Z \in \mathcal{N}_d$.

(iii) Let x be an element of $M(\Delta, d) \subset \partial D$ such that $d \geq 2$. Let U be a neighborhood of x in G. Suppose that U does not intersect $\bigcup_{\Delta \in \mathcal{B}} M(\Delta, 1)$. Then $U - \partial D$ is connected because $U \cap \partial D$ is a finite union of locally closed submanifolds of codimension greater than two. But $U \cap D$ and $U - D^{cl}$ are both non-empty open sets by Lemma 2.6, a contradiction. Thus $\partial D = (\bigcup_{\Delta \in \mathcal{B}} M(\Delta, 1))^{cl}$. $\qquad\qquad\square$

Remark 2.9. (i) It is known that $d \geq \dim(\mathfrak{s}_\Delta \cap \mathfrak{t})$ ([KR], Proposition 9). Hence

$$M(\Delta, 1) \neq \emptyset$$

only when the rank of $\Sigma_\Delta = \Sigma_\Delta^+ \sqcup \Sigma_\Delta^-$ is one.

(ii) In general we can prove that $\mathrm{codim}_G H'xH \geq \dim \mathfrak{t}$ for any $x = (\exp X_n)x_s \in G$. So we may call $x = (\exp X_n)x_s$ regular if $\mathrm{codim}_G H'xH = \dim \mathfrak{t}$. Regular semisimple elements studied in [M5] are contained in the set of regular elements.

The author is grateful to Michael Otto for suggesting the existence of this notion of regularity.

(iii) Suppose that $\mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha) \neq 0$ for all the longest roots α in Σ. Since all the roots in Σ are contained in the convex hull spanned by the longest roots, $\mathfrak{t}^+$ is written as

$$\mathfrak{t}^+ = \{Y \in \mathfrak{t} \mid |\alpha(Y)| < \frac{\pi}{2} \text{ for all the longest roots } \alpha \in \Sigma\}.$$

So the condition $M(\Delta, 1) \neq \emptyset$ for $\Delta \in \mathcal{B}$ implies that

$$\Sigma_\Delta = \Sigma_\Delta^- = \{\pm\alpha\}$$

for some longest root α. Suppose furthermore that all the longest roots in Σ are mutually $W_{K \cap H}(\mathfrak{t})$-conjugate. Then we can write

$$\bigcup_{\Delta \in \mathcal{B}} M(\Delta, 1) = \bigcup_{\Delta \in \mathcal{B}_\alpha} M(\Delta, 1)$$

with a longest root α in Σ, where

$$\mathcal{B}_\alpha = \{\Delta \in \mathcal{B} \mid \alpha(Y) = \frac{\pi}{2}i \text{ for all } Y \in \Delta\}.$$

So we have $\partial D = \left(\bigcup_{\Delta \in \mathcal{B}_\alpha} M(\Delta, 1)\right)^{cl}$.

3. Construction of parabolic subgroups

Let α be a root in $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t})$ and suppose that $2\alpha \notin \Sigma$. Let $\mathfrak{s}_{\mathbb{C}}$ be the subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha) \oplus \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, -\alpha)$. Since $[\mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha), \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, -\alpha)] \subset \mathfrak{z}_{\mathfrak{k}_{\mathbb{C}}}(\mathfrak{t})$ and since $[\mathfrak{z}_{\mathfrak{k}_{\mathbb{C}}}(\mathfrak{t}), \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \pm\alpha)] \subset \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \pm\alpha)$, we have

$$(3.1) \qquad\qquad \mathfrak{s}_{\mathbb{C}} \subset \mathfrak{z}_{\mathfrak{k}_{\mathbb{C}}}(\mathfrak{t}) \oplus \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha) \oplus \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, -\alpha).$$

Clearly $\mathfrak{s} = \mathfrak{s}_{\mathbb{C}} \cap \mathfrak{g}$ is a real form of $\mathfrak{s}_{\mathbb{C}}$.

Proposition 3.1. $(\mathfrak{s}, \mathfrak{s}^\tau) \cong (\mathfrak{so}(2, n), \mathfrak{so}(1, n))$, where $n = \dim_{\mathbb{C}} \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha)$.

Proof. Let $X \mapsto \overline{X}$ denote the conjugation with respect to the real form $\mathfrak{g}$. Since $X \mapsto \overline{\tau(X)}$ is a conjugation of $\mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha)$, we can take a non-zero element $X \in \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha)$ such that $\tau(X) = \overline{X}$. Put $Y_1 = [X, \tau(X)] = [X, \overline{X}]$. Then we have

$$Y_1 \in \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{t}) \cap \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{q}_{\mathbb{C}}' \cap i\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{t}) \cap i(\mathfrak{k} \cap \mathfrak{q}) = i\mathfrak{t}.$$

On the other hand, we have

$$B(Y, Y_1) = B(Y, [X, \overline{X}]) = B([Y, X], \overline{X}) = \alpha(Y)B(X, \overline{X})$$

for $Y \in \mathfrak{t}_{\mathbb{C}}$. Since the Hermitian form $B(X, \overline{X})$ is positive definite on $\mathfrak{m}_{\mathbb{C}}$, we see that $Y_1 \neq 0$. Taking $Y = Y_1$, we have

$$(3.2) \qquad\qquad\qquad\qquad \alpha(Y_1) > 0.$$

Since $Y_1 \in \mathfrak{s}_{\mathbb{C}}$ and since $[Y_1, \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \pm\alpha)] = \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \pm\alpha)$ by (3.2), the spaces $\mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \pm\alpha)$ are contained in the derived ideal $[\mathfrak{s}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}}]$ of $\mathfrak{s}_{\mathbb{C}}$. So we have $\mathfrak{s}_{\mathbb{C}} = [\mathfrak{s}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}}]$ and therefore $\mathfrak{s}_{\mathbb{C}}$ is semisimple. If $Z \in \mathfrak{s}_{\mathbb{C}} \cap \mathfrak{t}_{\mathbb{C}}$ satisfies $\alpha(Z) = 0$, then Z is contained in the center of $\mathfrak{s}_{\mathbb{C}}$ and hence $Z = 0$. So we have proved

$$(3.3) \qquad\qquad\qquad\qquad \mathfrak{s}_{\mathbb{C}} \cap \mathfrak{t}_{\mathbb{C}} = \mathbb{C}Y_1.$$

We will show that $\mathbb{C}Y_1$ is maximal abelian in $\mathfrak{s}_{\mathbb{C}} \cap \mathfrak{q}_{\mathbb{C}}'$. Let X be an element in $\mathfrak{s}_{\mathbb{C}} \cap \mathfrak{q}_{\mathbb{C}}'$ such that $[Y_1, X] = 0$. If $Y \in \mathfrak{t}$ satisfies $\alpha(Y) = 0$, then $[Y, X] = 0$ because $\mathfrak{s}_{\mathbb{C}}$ is generated by $\mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha) \oplus \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, -\alpha)$. Hence X is contained in the centralizer

of $\mathfrak{t}$. By (3.1) X is contained in $\mathfrak{z}_{\mathfrak{k}_{\mathbb{C}}}(\mathfrak{t})$. Hence $X \in \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{q}'_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{q}_{\mathbb{C}}$. Since $\mathfrak{t}$ is maximal abelian in $\mathfrak{k} \cap \mathfrak{q}$, we have $X \in \mathfrak{t}_{\mathbb{C}}$. It follows from (3.3) that $X \in \mathbb{C}Y_1$.

Thus the symmetric pair $(\mathfrak{s}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}} \cap \mathfrak{h}'_{\mathbb{C}})$ is rank one and we have the restricted root space decomposition

$$\mathfrak{s}_{\mathbb{C}} = \mathfrak{z}_{\mathfrak{s}_{\mathbb{C}}}(\mathfrak{t}) \oplus \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha) \oplus \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, -\alpha)$$

with respect to $\mathbb{C}Y_1$. We have such a restricted root space decomposition only when $(\mathfrak{s}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}} \cap \mathfrak{h}'_{\mathbb{C}}) \cong (\mathfrak{so}(n+2, \mathbb{C}), \mathfrak{so}(n+1, \mathbb{C}))$, where $n = \dim_{\mathbb{C}} \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha)$.

By the classification in [Be], we have

$$(\mathfrak{s}, \mathfrak{s}^\tau) \cong (\mathfrak{so}(p, q), \mathfrak{so}(p-1, q)) \qquad (q = n + 2 - p)$$

with some $p = 1, \ldots, n+1$ since $\mathfrak{s}$ is non-compact. (This can also be deduced from [M6] by considering two commuting involutions τ and θ on the compact real form $(\mathfrak{s} \cap \mathfrak{k}) \oplus i(\mathfrak{s} \cap \mathfrak{m}) \cong \mathfrak{so}(n+2)$.) If $p = 1$, then we have $\mathfrak{s} \cap \mathfrak{k} = \mathfrak{s} \cap \mathfrak{h}'$, a contradiction to $\mathbb{R}iY_1 = \mathfrak{s} \cap \mathfrak{t} \subset \mathfrak{k} \cap \mathfrak{q}'$. Hence $p \geq 2$. Taking $\mathfrak{s} \cap \mathfrak{t} = \mathbb{R}(E_{12} - E_{21}) \subset \mathfrak{s} \cap \mathfrak{q}'$, we can compute

$$\dim_{\mathbb{C}}(\mathfrak{s}_{\mathbb{C}}(\mathfrak{t}, \alpha) \cap \mathfrak{k}_{\mathbb{C}}) = p - 2.$$

Since $\mathfrak{s}_{\mathbb{C}}(\mathfrak{t}, \alpha) \subset \mathfrak{m}_{\mathbb{C}}$, we have $p = 2$. $\qquad \square$

For $Z \in \mathfrak{m}$ we define a parabolic subgroup P_Z of G by

$$P_Z = Z_G(Z) \exp \mathfrak{n}_Z,$$

where $\mathfrak{n}_Z = \bigoplus_{\lambda > 0} \{X \in \mathfrak{g} \mid [Z, X] = \lambda X\}$. For $\alpha \in \Sigma$ let Y_α denote the unique element in $\mathfrak{t}$ such that $\alpha(Y_\alpha) = \pi i/2$ and that $B(Y_\alpha, \mathfrak{t}_\alpha) = \{0\}$, where $\mathfrak{t}_\alpha = \{Y \in \mathfrak{t} \mid \alpha(Y) = 0\}$. Put $a_\alpha = \exp Y_\alpha$.

Theorem 3.2. *Suppose that $\mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha) \neq 0$ for all the longest roots α in Σ and that all the longest roots in Σ are mutually conjugate under $W_{K \cap H}(\mathfrak{t})$. Take a longest root α in Σ and a non-zero element Z of $(\mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha) \oplus \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, -\alpha)) \cap \mathfrak{h}'$. Then we have:*
(i) *$H'P_{\pm Z}$ and $Ha_\alpha^{-1}P_{\pm Z}$ are closed in G.*
(ii) *For any element x in the boundary of D, we have*

$$xHa_\alpha^{-1}P_Z \cap H'P_Z \neq \emptyset \qquad or \qquad xHa_\alpha^{-1}P_{-Z} \cap H'P_{-Z} \neq \emptyset.$$

(iii) *Assume moreover that $\dim \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha) \geq 2$ or that there exists a $t \in T \cap H$ such that $\mathrm{Ad}(t)Z = -Z$. Then for any element x in the boundary of D, we have*

$$xHa_\alpha^{-1}P_Z \cap H'P_Z \neq \emptyset.$$

Proof. (i) Since $Z \in \mathfrak{m} \cap \mathfrak{h}'$, it follows that $H' \cap P_Z$ is a parabolic subgroup of H'. Hence $H'P_Z/P_Z \cong H'/H' \cap P_Z$ is compact. This implies that $H'P_Z$ is closed in G. Replacing Z by $-Z$, we see that $H'P_{-Z}$ is also closed in G.

Identify $\mathfrak{s}$ with $\mathfrak{so}(2, n)$ as in Proposition 3.1. Then we have

$$\mathfrak{s} \cap \mathfrak{t} = \mathbb{R}(E_{21} - E_{12}),$$

where E_{jk} denote the matrix units. We can assume that $\alpha \in \Sigma$ satisfies $\alpha : y(E_{21} - E_{12}) \mapsto iy$. Then we have $Y_\alpha = \frac{\pi}{2}(E_{21} - E_{12})$ and therefore the adjoint action of $a_\alpha = \exp Y_\alpha$ on $\mathfrak{s}$ is equal to that of

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}$$

on $\mathfrak{so}(2, n)$ by the identification $\mathfrak{s} \cong \mathfrak{so}(2, n)$.

On the other hand, Z is of the form

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_1 & \cdots & x_n \\ 0 & x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_n & 0 & \cdots & 0 \end{pmatrix}$$

with some $(x_1, \ldots, x_n) \in \mathbb{R}^n - \{0\}$. Put $Z_0 = E_{23} + E_{32}$. Then we see that Z is $Z_{K \cap H}(\mathfrak{t})$-conjugate to rZ_0 with some $r \in \mathbb{R}^\times$. (We can take $r > 0$ if $n \geq 2$.) Since $P_{cZ} = P_Z$ for $c > 0$ and since we consider $\pm Z$, we may assume that

$$Z = Z_0.$$

We see that

$$a_\alpha^{-1} P_{\pm Z} a_\alpha = P_{\pm Z'},$$

where

$$Z' = \mathrm{Ad}(a_\alpha)^{-1} Z = E_{13} + E_{31} \in \mathfrak{s} \cap \mathfrak{m} \cap \mathfrak{q}' = \mathfrak{s} \cap \mathfrak{m} \cap \mathfrak{h}.$$

Hence $Ha_\alpha^{-1} P_{\pm Z} = HP_{\pm Z'} a_\alpha^{-1}$ are also closed in G by the same reason as for $H' P_{\pm Z}$.

(ii) If $h \in H$ and $h' \in H'$, then we have

$$(h'xh)Ha_\alpha^{-1} P_{\pm Z} \cap H' P_{\pm Z} = h'(xHa_\alpha^{-1} P_{\pm Z} \cap H' P_{\pm Z}).$$

So we may replace x by any element in the double coset $H'xH$.

By Remark 2.9(iii) we have $\partial D = (\bigcup_{\Delta \in \mathcal{B}_\alpha} M(\Delta, 1))^{cl}$. First assume that $x \in M(\Delta, 1)$ for some $\Delta \in \mathcal{B}_\alpha$. Then we have $x \in H'(\exp X_n)aH$ for some $a = \exp Y \in \exp \Delta$ and a nilpotent element X_n in $\mathfrak{q}' \cap \mathrm{Ad}(a)\mathfrak{q}$. By the above remark we may assume that

$$x = (\exp X_n)a.$$

By Lemma 2.7 we have

$$\mathfrak{g}_\mathbb{C}^{f_a} = \mathfrak{z}_{\mathfrak{k}_\mathbb{C}}(\mathfrak{t}) \oplus \mathfrak{m}_\mathbb{C}(\mathfrak{t}, \alpha) \oplus \mathfrak{m}_\mathbb{C}(\mathfrak{t}, -\alpha).$$

Since $\mathfrak{s}_\mathbb{C}$ is generated by $\mathfrak{m}_\mathbb{C}(\mathfrak{t}, \alpha) \oplus \mathfrak{m}_\mathbb{C}(\mathfrak{t}, -\alpha)$, it is contained in $\mathfrak{g}_\mathbb{C}^{f_a}$. Since

$$[\mathfrak{z}_{\mathfrak{k}_\mathbb{C}}(\mathfrak{t}), \mathfrak{s}_\mathbb{C}] \subset \mathfrak{s}_\mathbb{C},$$

$\mathfrak{s}_\mathbb{C}$ is an ideal of $\mathfrak{g}_\mathbb{C}^{f_a}$. Hence $\mathfrak{s} = \mathfrak{s}_\mathbb{C} \cap \mathfrak{g}$ is an ideal of $\mathfrak{g}^{f_a}$. Let $\mathfrak{s}^\perp$ denote the orthogonal complement of $\mathfrak{s}$ in $\mathfrak{g}^{f_a}$ with respect to the Killing form on $\mathfrak{g}$. Then $\mathfrak{s}^\perp$ is an ideal of $\mathfrak{g}^{f_a}$ contained in $\mathfrak{z}_\mathfrak{k}(\mathfrak{t})$. (Note that $\mathfrak{s}$ may be smaller than $\mathfrak{s}_\Delta$ in Section 2.)

Since $\mathfrak{g}^{f_a} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ and $\mathfrak{s}^\perp \subset \mathfrak{k}$, the nilpotent element X_n is contained in $\mathfrak{s} \cap \mathfrak{q}'$. By the identification in Proposition 3.1, X_n is of the form

$$\begin{pmatrix} 0 & -x_1 & x_2 & \cdots & x_{n+1} \\ x_1 & & & & \\ x_2 & & & & \\ \vdots & & & 0 & \\ x_{n+1} & & & & \end{pmatrix}$$

with $x_j \in \mathbb{R}$ such that $x_1^2 = x_2^2 + \cdots + x_{n+1}^2$. Since we can consider any element in the double coset $H'xH$, we can replace X_n by an $H' \cap aHa^{-1}$-conjugate. Since

$\mathfrak{h}' \cap \mathrm{Ad}(a)\mathfrak{h} = \mathfrak{g}^{f_a} \cap \mathfrak{h}' \supset \mathfrak{s} \cap \mathfrak{h}' \cong \mathfrak{so}(1,n)$, we may replace X_n by some $SO(1,n)_0$-conjugate by the identification in Proposition 3.1. So we may assume

$$X_n = \pm(E_{21} - E_{12} + \varepsilon E_{31} + \varepsilon E_{13}),$$

where $\varepsilon = \pm 1$. (We may put $\varepsilon = 1$ if $n \geq 2$.) By computation we see that

$$[Z, X_n] = \varepsilon X_n.$$

Since $\alpha(Y) = \alpha(Y_\alpha) = \pi i/2$, we can write $Y = Y_\alpha + Y'$ with $Y' \in \mathfrak{t} \cap \mathfrak{s}^\perp$. Hence we can write

$$a = a_\alpha a'$$

with $a' = \exp Y'$.

Since $X_n \in \mathfrak{n}_{\varepsilon Z}$ and since $Y' \in \mathfrak{s}^\perp \subset \mathfrak{z}_{\mathfrak{g}}(Z)$, it follows that

$$xHa_\alpha^{-1}P_{\varepsilon Z} \supset xa_\alpha^{-1}P_{\varepsilon Z} = (\exp X_n)a'P_{\varepsilon Z} = P_{\varepsilon Z}.$$

Hence we have

(3.4) $$xHa_\alpha^{-1}P_{\varepsilon Z} \cap H'P_{\varepsilon Z} \neq \emptyset$$

for $\varepsilon = 1$ or -1.

Finally let y be an arbitrary element in the boundary of D. If $yHa_\alpha^{-1}P_{\pm Z} \cap H'P_{\pm Z} = \emptyset$, then $xHa_\alpha^{-1}P_{\pm Z} \cap H'P_{\pm Z} = \emptyset$ for all the elements x in a neighborhood U_y of y because $yHa_\alpha^{-1}P_Z/P_Z$ and $H'P_Z/P_Z$ are compact. But this contradicts (3.4) because $\partial D = (\bigcup_{\Delta \in \mathcal{B}_\alpha} M(\Delta, 1))^{cl}$.

The assertion (iii) is also proved in (ii). $\square$

4. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. So we assume that G is a complex Lie group and that $H' (= G_{\mathbb{R}})$ is a real form of G.

Since K is a compact real form of G, we have

$$\dim_{\mathbb{C}} \mathfrak{m}_{\mathbb{C}}(\mathfrak{t}, \alpha) = \dim_{\mathbb{C}} \mathfrak{k}_{\mathbb{C}}(\mathfrak{t}, \alpha) = \dim_{\mathbb{C}} \mathfrak{g}(\mathfrak{t}, \alpha)$$

for all $\alpha \in \Sigma(\mathfrak{t})$. Hence we can identify $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t})$ with the usual restricted root system $\Sigma = \Sigma(\mathfrak{t})$ of H'. It is also known that all the longest roots in Σ are mutually conjugate under $W_{K \cap H}(\mathfrak{t})$ for simple H'. So the conditions in Theorem 3.2 are satisfied.

Table 1 gives the well-known list of simple real Lie algebras $\mathfrak{h}'$ and the multiplicities n of the longest restricted roots. (Assume $p \leq q$.)

In the following arguments we consider the complex structure only inside $\mathfrak{g}$. It means that we do not consider the "complexification" of the complex Lie algebra $\mathfrak{g}$ in order to avoid confusion.

We define a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m} \cap \mathfrak{q}$ by

$$\mathfrak{a} = \mathbb{R}Z \oplus i\mathfrak{t}_\alpha,$$

where $\mathfrak{t}_\alpha = \{Y \in \mathfrak{t} \mid \alpha(Y) = 0\}$. Take a positive system $\Sigma(\mathfrak{a})^+$ of the restricted root system $\Sigma(\mathfrak{a}) = \Sigma(\mathfrak{g}, \mathfrak{a})$ so that

$$\beta(Z) > 0 \text{ for all } \beta \in \Sigma(\mathfrak{a})^+.$$

Then the pair $(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ defines a parabolic subgroup

$$P = P(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$$

of G contained in P_Z.

TABLE 1.

type	$\mathfrak{h}'$	Σ	n	
AI	$\mathfrak{sl}(\ell, \mathbb{R})$	$A_{\ell-1}$	1	
AII	$\mathfrak{gl}(\ell, \mathbb{H})/\mathbb{R}$	$A_{\ell-1}$	4	
AIII	$\mathfrak{su}(p, q)$	BC_p or C_p	1	Hermitian
BDI	$\mathfrak{so}(p, q)$ $(p \geq 2)$	B_p or D_p	1	Hermitian if $p = 2$
BDI	$\mathfrak{so}(1, q)$	B_1	$q-1$	
CI	$\mathfrak{sp}(\ell, \mathbb{R})$	C_ℓ	1	Hermitian
CII	$\mathfrak{sp}(p, q)$	BC_p or C_p	3	
DIII	$\mathfrak{so}^*(4\ell)$	C_ℓ	1	Hermitian
DIII	$\mathfrak{so}^*(4\ell + 2)$	BC_ℓ	1	Hermitian
EI		E_6	1	
EII		F_4	1	
EIII		BC_2	1	Hermitian
EIV		A_2	8	
EV		E_7	1	
EVI		F_4	1	
EVII		C_3	1	Hermitian
EVIII		E_8	1	
EIX		F_4	1	
FI		F_4	1	
FII		BC_1	7	
G		G_2	1	
complex cases			2	

Let $\mathfrak{j}$ be a maximal abelian subspace of $\mathfrak{m}$ containing $\mathfrak{a}$. Let $\Sigma(\mathfrak{j})^+$ be a positive system of the root system $\Sigma(\mathfrak{j}) = \Sigma(\mathfrak{g}, \mathfrak{j})$ which is compatible with $\Sigma(\mathfrak{a})^+$. Then the pair $(\mathfrak{j}, \Sigma(\mathfrak{j})^+)$ defines a Borel subgroup

$$B = B(\mathfrak{j}, \Sigma(\mathfrak{j})^+)$$

of G contained in P. Since $\mathfrak{a} = \mathfrak{j} \cap \mathfrak{q}$ is maximal abelian in $\mathfrak{m} \cap \mathfrak{q}$ and since $\Sigma(\mathfrak{j})^+$ is τ-compatible, it follows that $H'B$ is closed in G and that HB is open in G ([M1], Proposition 1 and Proposition 2). Since H is a complex symmetric subgroup of G, HB is the unique open H-B double coset in G and therefore $H'B$ is the unique closed H'-B double coset in G.

Remark 4.1. Since Theorem 1.3 concerns orbits on the flag manifold $X = G/B$, we may replace B with any conjugate pBp^{-1} for $p \in P$.

Let x be an element in the boundary of D. By Theorem 3.2 we have

$$(4.1) \qquad\qquad xHa_\alpha^{-1}P_Z \cap H'P_Z \neq \emptyset$$

or

$$(4.2) \qquad\qquad xHa_\alpha^{-1}P_{-Z} \cap H'P_{-Z} \neq \emptyset.$$

If $G_\mathbb{R}$ is of non-Hermitian type, then the condition in Theorem 3.2(iii) is satisfied by Lemma 7.2 in the appendix. Hence we have (4.1) in this case.

Assume (4.1). Then we will show that

$$(4.3) \qquad\qquad x(Ha_\alpha^{-1}pB)^{cl} \cap H'B \neq \emptyset$$

for any $p \in P$. It follows from Lemma 7.1(i) that $HP = HB$. So we have $H'P = H'B$ ([M2]). Hence (4.3) is equivalent to

$$x(Ha_\alpha^{-1}P)^{cl} \cap H'P \neq \emptyset$$

because $(Ha_\alpha^{-1}P)^{cl} = (Ha_\alpha^{-1}pB)^{cl}P$. We have only to show that

$$(Ha_\alpha^{-1}P)^{cl} = Ha_\alpha^{-1}P_Z$$

which is equivalent to $(HP')^{cl} = HP_{Z'}$, where $P' = a_\alpha^{-1}Pa_\alpha$. Put

(4.4) $$\mathfrak{a}' = \mathrm{Ad}(a_\alpha^{-1})\mathfrak{a} = \mathbb{R}Z' \oplus i\mathfrak{t}_\alpha.$$

Then the Lie algebra $\mathfrak{p}'$ of P' is defined by the pair $(\mathfrak{a}', \Sigma(\mathfrak{a}')^+)$, where $\Sigma(\mathfrak{a}')^+ = \{\alpha \circ \mathrm{Ad}(a_\alpha) \in \Sigma(\mathfrak{a}') \mid \alpha \in \Sigma(\mathfrak{a})^+\}$. Since H and P' are complex subgroups of G, we have only to show the equality

(4.5) $$\mathfrak{h} + \mathfrak{p}' = \mathfrak{h} + \mathfrak{p}_{Z'}.$$

We can write

$$\mathfrak{p}_{Z'} = \mathfrak{p}' \oplus \bigoplus_{\beta \in \Sigma(\mathfrak{a}')^+, \beta(Z')=0} \mathfrak{g}(\mathfrak{a}', -\beta).$$

If $\beta \in \Sigma(\mathfrak{a}')^+$ satisfies $\beta(Z') = 0$, then it follows from (4.4) that $\sigma\beta = -\beta$ and hence $\sigma\mathfrak{g}(\mathfrak{a}', -\beta) = \mathfrak{g}(\mathfrak{a}', \beta)$. So we have $\mathfrak{p}_{Z'} \subset \mathfrak{p}' + \sigma\mathfrak{p}' \subset \mathfrak{h} + \mathfrak{p}'$. (4.5) is clear by this inclusion. Thus we have proved (4.3).

Next assume (4.2). Since $\mathrm{Ad}(a_\alpha^2)Z = -Z$, we have $P_{-Z} = a_\alpha^2 P_Z a_\alpha^{-2}$. Hence

$$xHa_\alpha^{-1}a_\alpha^2 P_Z a_\alpha^{-2} \cap H'a_\alpha^2 P_Z a_\alpha^{-2} \neq \emptyset$$

and therefore

$$xHa_\alpha P_Z \cap H'a_\alpha^2 P_Z \neq \emptyset.$$

Since $H'P_Z$ and $H'a_\alpha^2 P_Z = H'P_{-Z}a_\alpha^2$ are closed in G and since there is only one closed H'-P_Z double coset in G, we have $H'P_Z = H'a_\alpha^2 P_Z$. Hence

$$xHa_\alpha P_Z \cap H'P_Z \neq \emptyset.$$

By the same argument as for (4.1) $\Longrightarrow$ (4.3), we get

$$x(Ha_\alpha p'B)^{cl} \cap H'B \neq \emptyset$$

for an arbitrary $p' \in P$. Thus we have proved (ii) for $\widetilde{S} = Ha_\alpha^{-1}pB \cup Ha_\alpha p'B$ or $\widetilde{S} = Ha_\alpha^{-1}pB$ for arbitrary p and p' in P.

Remark 4.2. In the following proof of (iii), we must choose $p \in P$ (resp. $p' \in P$) so that $Ha_\alpha^{-1}pB$ (resp. $Ha_\alpha p'B$) is as small as possible. Of course we may also replace B with pBp^{-1} for some $p \in P$. The following considerations on orbit structure are based on many examples computed in [MO], Section 4.

Now we will prove (iii). First consider the case where n is odd. Then in view of Proposition 3.1 and the choice of Z in the proof of Theorem 3.2, we can take a maximal abelian subalgebra $\mathfrak{t}_\mathfrak{s}$ of $\mathfrak{s} \cap \mathfrak{k} \cap \mathfrak{h}(\subset \mathfrak{z}_\mathfrak{g}(\mathfrak{t}))$ so that $[Z, \mathfrak{t}_\mathfrak{s}] = \{0\}$. We may choose $\mathfrak{j}$ so that it contains $\mathfrak{a} \oplus i\mathfrak{t}_\mathfrak{s}$. We see that $\mathrm{Ad}(a_\alpha^{-2})$ defines the reflection with respect to Z on $\mathfrak{j} \cap \mathfrak{s}_\mathbb{C}$. Since $\mathrm{Ad}(a_\alpha^{-2})$ acts trivially on $\mathfrak{s}_\mathbb{C}^\perp$, it acts on $\mathfrak{j}$ as the reflection w_Z with respect to Z.

Let Ψ denote the set of simple roots in $\Sigma(\mathfrak{j})^+$. For each subset Θ of Ψ there corresponds a parabolic subgroup

$$P_\Theta = BW_\Theta B$$

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of G, where W_Θ is the subgroup of W generated by $\{w_\beta \mid \beta \in \Theta\}$. We will show that

$$\Theta \neq \Psi \implies H'a_\alpha^{-1}B \cap H'P_\Theta = \emptyset$$

because the right-hand side is equivalent to $Ha_\alpha^{-1}B \cap HP_\Theta = \emptyset$ ([M2]).

We can study H'-B double cosets by using the Bruhat decomposition as in [Sp]. By the map

$$y \mapsto \tau(y)^{-1}y,$$

the double coset $H'yB$ is mapped into $\tau(B)\tau(y)^{-1}yB$. Since $\Sigma(\mathfrak{j})^+$ is τ-compatible, we can write

$$\tau(B) = w_c^{-1}Bw_c,$$

where w_c is the longest element in the Weyl group W_c generated by the reflections with respect to the compact roots in $\Sigma(\mathfrak{j})$. Hence by the map

$$y \mapsto w_c\tau(y)^{-1}y,$$

$H'yB$ is mapped into

$$w_c\tau(B)\tau(y)^{-1}yB = Bw_c\tau(y)^{-1}yB.$$

By this map $H'a_\alpha^{-1}B$ is mapped into $Bw_ca_\alpha^{-2}B$. On the other hand, $H'P_\Theta$ is mapped into

$$w_c\tau(P_\Theta)w_c^{-1}w_cP_\Theta = P_{\Theta'}w_cP_\Theta.$$

Here we write $w_c\tau(P_\Theta)w_c^{-1} = P_{\Theta'}$ with some $\Theta' \subset \Psi$ since $w_c\tau(P_\Theta)w_c^{-1}$ contains $w_c\tau(B)w_c^{-1} = B$. Since $w_ca_\alpha^{-2}$ normalizes $\mathfrak{j}$, it is considered an element of W. So we have only to prove that

$$(4.6) \qquad\qquad\qquad w_ca_\alpha^{-2} \notin W_{\Theta'}W_cW_\Theta.$$

Let Z_Θ (resp. $Z_{\Theta'}$) be a non-zero element in $\mathfrak{j}$ which is dominant with respect to $\Sigma(\mathfrak{j})^+$ and W_Θ-invariant (resp. $W_{\Theta'}$-invariant). Then we have

$$(4.7) \qquad\qquad\qquad B(Z, Z_\Theta) > 0 \text{ (resp. } B(Z, Z_{\Theta'}) > 0)$$

with respect to the Killing form $B(\ ,\ )$ as follows. Z is identified by $B(\ ,\ )$ with a dominant root $\widetilde{\alpha}$ in $\Sigma(\mathfrak{j})^+$ (modulo positive constant). It is known that $\widetilde{\alpha} = \sum_{\beta \in \Psi} m_\beta \beta$ with $m_\beta > 0$ for all $\beta \in \Psi$. Since $\beta(Z_\Theta) \geq 0$ for all $\beta \in \Psi$ and since $\beta(Z_\Theta) > 0$ for some $\beta \in \Psi$, it follows that

$$\widetilde{\alpha}(Z_\Theta) = \sum_{\beta \in \Psi} m_\beta \beta(Z_\Theta) > 0.$$

Since $Z \in \mathfrak{a}$, we have

$$(4.8) \qquad\qquad\qquad\qquad w_c(Z) = Z.$$

Since Z_Θ and $Z_{\Theta'}$ are dominant for $\Sigma(\mathfrak{j})^+$, we have

$$(4.9) \qquad\qquad B(Z_{\Theta'}, \mathrm{Ad}(w_c)Z_\Theta) \leq B(Z_{\Theta'}, \mathrm{Ad}(w_1)Z_\Theta)$$

for any $w_1 \in W_c$.

It follows from (4,7), (4.8) and (4.9) that

$$B(Z_{\Theta'}, \mathrm{Ad}(w_c)\mathrm{Ad}(a_\alpha^{-2})Z_\Theta) = B(Z_{\Theta'}, \mathrm{Ad}(w_c)(Z_\Theta - \frac{2B(Z, Z_\Theta)}{B(Z, Z)} Z))$$

$$= B(Z_{\Theta'}, \mathrm{Ad}(w_c)Z_\Theta) - \frac{2B(Z, Z_\Theta)B(Z, Z_{\Theta'})}{B(Z, Z)}$$

$$< B(Z_{\Theta'}, \mathrm{Ad}(w_1)Z_\Theta)$$

$$= B(Z_{\Theta'}, \mathrm{Ad}(w_{\Theta'}w_1 w_\Theta)Z_\Theta)$$

for all $w_\Theta \in W_\Theta$, $w_1 \in W_c$ and $w_{\Theta'} \in W_{\Theta'}$. Thus we have (4.6).
We can also prove that

$$\Theta \neq \Psi \Longrightarrow H'a_\alpha B \cap H'P_\Theta = \emptyset$$

by the same argument. Thus we have proved (iii) when n is odd.

5. Proof of (iii) for even n

In this section we will prove (iii) of Theorem 1.3 when n is even. In this case there are no real roots in $\Sigma(\mathfrak{j})$ and hence

(5.1)   there is only one $K \cap H$-conjugacy class
of σ-stable maximal abelian subspace of $\mathfrak{m}$

([M1], Theorem 2, [Su], Theorem 6).
By the classification, there are four cases

$$H' (= G_\mathbb{R}) \text{ is complex,} \quad \mathrm{AII}, \quad \mathfrak{so}(1, q) \ (q \text{ is odd}) \text{ and } \mathrm{EIV}.$$

5.1. **Complex cases.** Let G_1 be a complex simple Lie group. Let K_1 be a compact real form of G_1 and let θ_1 be the conjugation of G_1 with respect to K_1. Then $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{m}_1$ is a Cartan decomposition of $\mathfrak{g}_1$, where $\mathfrak{m}_1 = i\mathfrak{k}_1$.
Put $G = G_1 \times G_1$, $H = \{(g, g) \mid g \in G_1\}$ and $K = K_1 \times K_1$. Then

$$H' = \{(g, \theta_1(g)) \mid g \in G_1\}.$$

Let $\mathfrak{a} = \mathbb{R}Z \oplus i\mathfrak{t}_\alpha \subset \mathfrak{m} \cap \mathfrak{q}$ be as in Section 4. Since $\mathfrak{m} \cap \mathfrak{q} = \{(X, -X) \mid X \in \mathfrak{m}_1\}$, we can write $\mathfrak{a} = \{(X, -X) \mid X \in \mathfrak{j}_1\}$ with some maximal abelian subspace $\mathfrak{j}_1$ of $\mathfrak{m}_1$.
Write $Z = (Z_1, -Z_1)$, $a_\alpha = (a, a^{-1}) = (\exp Y, \exp(-Y))$, $i\mathfrak{t}_\alpha = \{(X, -X) \mid X \in (\mathfrak{j}_1)_\alpha\}$ and

$$Z' = \mathrm{Ad}(a_\alpha^{-1})Z = (\mathrm{Ad}(a^{-1})Z_1, -\mathrm{Ad}(a)Z_1) = (Z_2, Z_2) \in \mathfrak{m} \cap \mathfrak{h}$$

with $Z_1 \in \mathfrak{j}_1$, $Y \in \mathfrak{k}_1$ and $Z_2 \in \mathfrak{m}_1$. Then we have

$$\mathrm{Ad}(a^2)Z_1 = -\mathrm{Ad}(a)Z_2 = -Z_1 \quad \text{and} \quad \mathrm{Ad}(a^2)X = X \text{ for } X \in (\mathfrak{j}_1)_\alpha.$$

Hence $\mathrm{Ad}(a^2)|_{\mathfrak{j}_1}$ is the reflection with respect to Z_1.
Let Σ_1 denote the root system of the pair $(\mathfrak{g}_1, \mathfrak{j}_1)$ and let Σ_1^+ be a positive system of Σ_1 such that Z_1 is dominant for Σ_1^+. Let B_1 be the Borel subgroup of G_1 for the pair $(\mathfrak{j}_1, \Sigma_1^+)$ and put $B = B_1 \times \theta_1(B_1)$. By the map

(5.2) $$H'(y, z)B \mapsto B_1 y^{-1}\theta_1(z)B_1$$

the decomposition $H'\backslash G/B$ is identified with the Bruhat decomposition $B_1\backslash G_1/B_1$ of G_1. Let Ψ_1 denote the set of simple roots in Σ_1^+. Then every parabolic subgroup P_Θ of G containing B is written as

$$P_\Theta = P_{\Theta_1} \times \theta_1(P_{\Theta_2})$$

with some subsets Θ_1 and Θ_2 of Ψ_1, where $P_{\Theta_i} = B_1 W_{\Theta_i} B_1$ is the parabolic subgroup of G_1 corresponding to Θ_i.

By the identification (5.2) $H'a_\alpha^{-1}B$ and $H'P_\Theta$ are identified with $B_1 a\theta_1(a)B_1 = B_1 a^2 B_1 = B_1 w_{Z_1} B_1$ and $P_{\Theta_1}P_{\Theta_2} = B_1 W_{\Theta_1} W_{\Theta_2} B_1$, respectively. Since we assume that $H'\backslash G/P_\Theta$ is non-trivial, Θ_1 and Θ_2 are not equal to Ψ_1. So we have only to show that

$$w_{Z_1} \notin W_{\Theta_1} W_{\Theta_2}.$$

But we can prove this by the same argument as in Section 4 since Z_1 corresponds to the maximal root in Σ_1^+. Thus we have proved (iii) when H' is a complex Lie group.

5.2. AII-case. Since $n = 4$, the pair $(\mathfrak{s}, \mathfrak{s}^\tau)$ is isomorphic to $(\mathfrak{so}(2,4), \mathfrak{so}(1,4))$ by Proposition 3.1. By this identification, we can take a three-dimensional maximal abelian subspace

$$\mathfrak{j}_\mathfrak{s} = \mathbb{R}Z \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$$

in $\mathfrak{m} \cap \mathfrak{s}_\mathbb{C}$, where $Z = E_{23} + E_{32}$ (as in Section 3), $Z_1 = E_{14} + E_{41}$ and $Z_2 = i(E_{56} - E_{65})$. Take a maximal abelian subspace $\mathfrak{j}_{\mathfrak{s}\perp}$ of $\mathfrak{m} \cap (\mathfrak{s}^\perp)_\mathbb{C} = i\mathfrak{s}^\perp$ containing $i\mathfrak{t}_\alpha$. Then $\mathfrak{j} = \mathfrak{j}_\mathfrak{s} \oplus \mathfrak{j}_{\mathfrak{s}\perp}$ is a maximal abelian subspace of $\mathfrak{m}$ containing $\mathfrak{a} = \mathbb{R}Z \oplus i\mathfrak{t}_\alpha$. By computation we have

$$\mathrm{Ad}(a_\alpha^{-2})Z = -Z, \quad \mathrm{Ad}(a_\alpha^{-2})Z_1 = -Z_1, \quad \mathrm{Ad}(a_\alpha^{-2})Z_2 = Z_2$$

and $\mathrm{Ad}(a_\alpha^{-2})X = X$ for all $X \in \mathfrak{j}_{\mathfrak{s}\perp}$.

Let $\Sigma(\mathfrak{a})^+$ be a positive system of $\Sigma(\mathfrak{a})$ such that Z is dominant for $\Sigma(\mathfrak{a})^+$. Let $\Sigma(\mathfrak{j})^+$ be a positive system of $\Sigma(\mathfrak{j})$ which is compatible with $\Sigma(\mathfrak{a})^+$. Let $\Psi = \{\alpha_1, \ldots, \alpha_m\}$ (m is odd) denote the set of simple roots in $\Sigma(\mathfrak{j})^+$, where

$$(\alpha_j, \alpha_k) = \begin{cases} 1 & \text{if } k = j, \\ -1/2 & \text{if } |k - j| = 1, \\ 0 & \text{if } |k - j| \geq 2 \end{cases}$$

as usual. Then $\{\alpha_k \mid k \text{ is odd}\}$ is the set of the compact simple roots in Ψ. Let $\alpha_{\max} = \alpha_1 + \cdots + \alpha_m$ denote the maximal root in $\Sigma(\mathfrak{j})^+$. Then the four roots

$$\alpha_{\max}, \quad \alpha_{\max} - \alpha_1, \quad \alpha_{\max} - \alpha_m \quad \text{and} \quad \alpha_{\max} - \alpha_1 - \alpha_m$$

are mapped onto the maximal root $\widetilde{\alpha}$ in $\Sigma(\mathfrak{a})^+$ by the restriction of roots in $\Sigma(\mathfrak{j})$ to $\mathfrak{a}$. By the Killing form we can identify Z and Z_1 with $\widetilde{\alpha}$ and a vector in $\mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_m$, respectively.

We see that

$$P_Z = P_{\Psi - \{\alpha_2, \alpha_{m-1}\}} = P_{\{\alpha_1\}} P_{\{\alpha_m\}} P_{\Theta_0},$$

where $\Theta_0 = \{\alpha_3, \alpha_4, \ldots, \alpha_{m-2}\}$. Since $Ha_\alpha^{-1}P_Z$ is closed in G by Theorem 3.2(i), it follows that

$$Ha_\alpha^{-1}pP_{\Theta_0}$$

is closed in G for some $p \in P_{\{\alpha_1\}} P_{\{\alpha_m\}}$. It follows from (5.1) and Lemma 7.1 that $Ha_\alpha^{-1}pB = Ha_\alpha^{-1}wB$ for some $w \in W_{\{\alpha_1\}} W_{\{\alpha_m\}} = \{e, w_{\alpha_1}, w_{\alpha_m}, w_{\alpha_1}w_{\alpha_m}\}$. (More precisely, we can see that $Ha_\alpha^{-1}B = Ha_\alpha^{-1}w_{\alpha_1}w_{\alpha_m}B$ and $Ha_\alpha^{-1}w_{\alpha_1}B =$

$Ha_\alpha^{-1}w_{\alpha_m}B$. But we do not need these equalities here.) Replace B by wBw^{-1}. Then

$$Ha_\alpha^{-1}P_{\Theta_0}$$

is closed in G.

First suppose that Θ contains Θ_0. If $Ha_\alpha^{-1}B \subset HP_\Theta$, then HP_Θ contains a closed set $Ha_\alpha^{-1}P_{\Theta_0}$. Hence HP_Θ is closed in G. But HP_Θ is also open in G. So we have $HP_\Theta = G$ contradicting the assumption $HP_\Theta \neq G$. Thus we have

$$Ha_\alpha^{-1}B \cap HP_\Theta = \emptyset.$$

So we have only to consider $\Theta = \Psi - \{\alpha_k\}$ with $k = 3, 4, \ldots, m - 2$. Take a Z_Θ as in Section 4. Since $w_c\tau(P_\Theta)w_c^{-1} = P_\Theta$, we have only to prove (4.6) for $\Theta' = \Theta$. Since $\alpha_1(Z_\Theta) = \alpha_m(Z_\Theta) = 0$, we have

(5.3)
$$w_{Z_1}Z_\Theta = Z_\Theta.$$

Since Z corresponds to a positive constant multiple of $2\alpha_{\max} - \alpha_1 - \alpha_m$ and since

$$(2\alpha_{\max} - \alpha_1 - \alpha_m)(Z_\Theta) = 2\alpha_{\max}(Z_\Theta) > 0$$

as in Section 4, we have

(5.4)
$$B(Z, Z_\Theta) > 0.$$

By (4.8), (4.9), (5.3) and (5.4) we have

$$\begin{aligned}
B(Z_\Theta, \mathrm{Ad}(w_c a_\alpha^{-2})Z_\Theta) &= B(Z_\Theta, \mathrm{Ad}(w_c w_Z w_{Z_1})Z_\Theta) \\
&= B(Z_\Theta, \mathrm{Ad}(w_c w_Z)Z_\Theta) \\
&= B(Z_\Theta, \mathrm{Ad}(w_c)(Z_\Theta - \frac{2B(Z, Z_\Theta)}{B(Z, Z)}Z)) \\
&= B(Z_\Theta, \mathrm{Ad}(w_c)Z_\Theta) - \frac{2B(Z, Z_\Theta)^2}{B(Z, Z)} \\
&< B(Z_\Theta, \mathrm{Ad}(w_1)Z_\Theta) \\
&= B(Z_\Theta, \mathrm{Ad}(w_\Theta w_1 w_\Theta')Z_\Theta)
\end{aligned}$$

for all $w_\Theta, w_\Theta' \in W_\Theta$ and $w_1 \in W_c$. Thus we have proved

$$w_c a_\alpha^{-2} \notin W_\Theta W_c W_\Theta.$$

5.3. Cases of $\mathfrak{so}(1,q)$ (q is odd) and EIV. In these cases, we have $P_Z = P$. (Remark: We only use this condition. So the following argument is also valid when H' ($= G_\mathbb{R}$) is real rank one.) Since $Ha_\alpha^{-1}P = Ha_\alpha^{-1}P_Z$ is closed in G by Theorem 3.2(i), there exists a $p \in P$ such that $Ha_\alpha^{-1}pB$ is closed in G. If $Ha_\alpha^{-1}pB \subset HP_\Theta$, then HP_Θ is closed in G. But HP_Θ is also open in G. Hence we have $HP_\Theta = G$, a contradiction to the assumption $HP_\Theta \neq G$. Thus we have proved

$$Ha_\alpha^{-1}pB \cap HP_\Theta = \emptyset$$

and we have completed the proof of Theorem 1.3.

6. Proof of Theorem 1.7

Applying Lemma 2 in [GM2] recursively, we can find a sequence of simple roots $\alpha_1, \ldots, \alpha_\ell$ ($\ell = \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} \widetilde{S}$) such that

$$\dim_{\mathbb{C}} \widetilde{S} P_{\alpha_1} \cdots P_{\alpha_k} = \dim_{\mathbb{C}} \widetilde{S} P_{\alpha_1} \cdots P_{\alpha_{k-1}} + 1$$

for $k = 1, \ldots, \ell$.

Lemma 6.1. *Suppose that H' is of non-Hermitian type.*
(i) *If H' is not of* CII-*type or* FII-*type, then*

$$(\widetilde{S}')^{cl} = S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_1}.$$

(ii) *If H' is of* CII-*type, then we can take α_1 as a compact root and we have*

$$S'_1{}^{cl} = S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_2}$$

for the dense H-B double coset S_1 in $\widetilde{S} P_{\alpha_1}$.

Proof. Let S_k denote the dense H-B double coset in $\widetilde{S} P_{\alpha_1} \cdots P_{\alpha_k}$ for $k = 0, \ldots, \ell$. Write $S_k = H x_k B$ with a representative x_k such that $\mathfrak{j}_k = \mathrm{Ad}(x_k)\mathfrak{j}$ is a σ-stable maximal abelian subspace in $\mathfrak{m}$. Let β_k denote the simple root in

$$\Sigma(\mathfrak{j}_k)^+ = \{\gamma \circ \mathrm{Ad}(x_k)^{-1} \mid \gamma \in \Sigma(\mathfrak{j})^+\}$$

defined by $\beta_k = \alpha_k \circ \mathrm{Ad}(x_k)^{-1}$. Since $\dim_{\mathbb{C}} S_k = \dim_{\mathbb{C}} S_{k-1} + 1$, it follows from Lemma 7.1 that

$$\dim(\mathfrak{j}_k \cap \mathfrak{q}) = \dim(\mathfrak{j}_{k-1} \cap \mathfrak{q}) \quad \text{and} \quad \beta_k \text{ is complex}$$

or that

$$\dim(\mathfrak{j}_k \cap \mathfrak{q}) = \dim(\mathfrak{j}_{k-1} \cap \mathfrak{q}) + 1 \quad \text{and} \quad \beta_k \text{ is real}.$$

In particular, we have

(6.1) $$\dim(\mathfrak{j}_0 \cap \mathfrak{q}) \leq \dim(\mathfrak{j}_1 \cap \mathfrak{q}) \leq \cdots \leq \dim(\mathfrak{j}_\ell \cap \mathfrak{q}).$$

Suppose that β_k is complex. Then it follows from Lemma 7.1(iv) that

$$S_k P_{\alpha_k} = S_{k-1} \sqcup S_k.$$

Moreover we can take $x_k = x_{k-1} w_{\alpha_k}$ and hence $\mathfrak{j}_k = \mathfrak{j}_{k-1}$. By the duality ([M2]) we have $S'_k P_{\alpha_k} = S'_{k-1} \sqcup S'_k$ and $S'_k \subset S'_{k-1}{}^{cl}$. Hence

(6.2) $$S'_{k-1}{}^{cl} = (S'_k P_{\alpha_k})^{cl} = S'_k{}^{cl} P_{\alpha_k}.$$

(i) First suppose that n is even. By (5.1) we have

$$\dim(\mathfrak{j}_0 \cap \mathfrak{q}) = \dim(\mathfrak{j}_1 \cap \mathfrak{q}) = \cdots = \dim(\mathfrak{j}_\ell \cap \mathfrak{q}).$$

Since β_k is complex for all $k = 1, \ldots, \ell$, it follows from (6.2) that

$$(\widetilde{S}')^{cl} = S'_0{}^{cl} = S'_1{}^{cl} P_{\alpha_1} = \cdots = S'_\ell{}^{cl} P_{\alpha_\ell} \cdots P_{\alpha_1} = S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_1}.$$

Next consider the case where n is odd. Since we assume H' is not of CII-type or FII-type, it follows that $n = 1$. Take a σ-stable maximal abelian subspace $\mathfrak{j}$ of $\mathfrak{m}$ as in Section 4 and Section 7.2. Then $\mathfrak{j}' = \mathrm{Ad}(a_\alpha^{-1})\mathfrak{j}$ is σ-stable. So we may assume that $x_0 = a_\alpha^{-1}$. Since $\mathfrak{j}' \cap \mathfrak{q} = i\mathfrak{t}_\alpha$, it follows from (6.1) that

$$\dim(\mathfrak{j}_0 \cap \mathfrak{q}) = \cdots = \dim(\mathfrak{j}_{m-1} \cap \mathfrak{q}) = \dim(\mathfrak{j}_m \cap \mathfrak{q}) - 1 = \cdots = \dim(\mathfrak{j}_\ell \cap \mathfrak{q}) - 1$$

for some m. So we may assume that $\mathfrak{j}_0 = \cdots = \mathfrak{j}_{m-1} = \mathfrak{j}'$ and that $\mathfrak{j}_m = \cdots = \mathfrak{j}_\ell = \mathfrak{j}$. We see that the root $\widetilde{\alpha} \in \Sigma(\mathfrak{j})$ corresponding to Z is the real root defining $\mathfrak{j}' =$

$\mathrm{Ad}(a_\alpha^{-1})\mathfrak{j}$ from $\mathfrak{j}$ ([Su], Theorem 6). On the other hand β_m is also a real root defining the $K \cap H$-conjugacy class of σ-stable maximal abelian subspace containing $\mathfrak{j}'$. So $\widetilde{\alpha}$ and β_m are conjugate under some $w \in W_{K\cap H}(\mathfrak{j})$. Replacing x_m by wx_m, we may assume that $\beta_m = \widetilde{\alpha}$. Consider the root $\gamma = \alpha_m \circ \mathrm{Ad}(x_{m-1})^{-1} = \beta_m \circ \mathrm{Ad}(x_m x_{m-1}^{-1})$ in $\Sigma(\mathfrak{j}_{m-1})$. This root γ is the non-compact simple root in $\Sigma(\mathfrak{j}_{m-1})^+$ corresponding to Z'.

Since H' is of non-Hermitian type, it follows from Lemma 7.2 that $Hw_\gamma x_{m-1}B = Hx_{m-1}B$. Hence we have

$$S_{m-1}P_{\alpha_m} = Hx_{m-1}B \sqcup Hc_\gamma x_{m-1}B = S_{m-1} \sqcup S_m$$

by Lemma 7.1(ii). So we have $S'_m P_{\alpha_m} = S'_m \sqcup S'_{m-1}$ and $S'_m \subset S'_{m-1}{}^{cl}$. Hence

(6.3) $$S'_{m-1}{}^{cl} = (S'_m P_{\alpha_m})^{cl} = S'_m{}^{cl} P_{\alpha_m}.$$

By (6.2) and (6.3), we get

$$(\widetilde{S}')^{cl} = S'_0{}^{cl} = S'_1{}^{cl} P_{\alpha_1} = \cdots = S'_\ell{}^{cl} P_{\alpha_\ell} \cdots P_{\alpha_1} = S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_1}.$$

(ii) Take $\mathfrak{j}$, $\Sigma(\mathfrak{j})^+$ and Ψ (the set of the simple roots in $\Sigma(\mathfrak{j})^+$) as in Section 4. Then $\mathfrak{j}_0 = \mathrm{Ad}(a_\alpha^{-1})\mathfrak{j}$ is σ-stable and $\dim(\mathfrak{j}_0 \cap \mathfrak{q}) = \dim(\mathfrak{j} \cap \mathfrak{q}) - 1$. Write $\Psi = \{e_1 - e_2, e_2 - e_3, \ldots, e_{r-1} - e_r, 2e_r\}$ as usual by using an orthonormal basis $\{e_1, \ldots, e_r\}$ of $\mathfrak{j}^*$. Then the compact simple roots are

$$e_1 - e_2,\ e_3 - e_4,\ \ldots,\ e_{2s-1} - e_{2s},\ e_{2s+1} - e_{2s+2},\ e_{2s+2} - e_{2s+3},\ \ldots, e_{r-1} - e_r,\ 2e_r,$$

where s $(\le r/2)$ is the real rank of $H' = Sp(s, r-s)$. The dominant root $\widetilde{\alpha}$ corresponding to Z is

$$\widetilde{\alpha} = e_1 + e_2.$$

The root $\beta = \widetilde{\alpha} \circ \mathrm{Ad}(a_\alpha) \in \Sigma(\mathfrak{j}_0)$ is non-compact. Suppose that $(e_1 - e_2) \circ \mathrm{Ad}(a_\alpha)$ is compact. Then $2e_1 \circ \mathrm{Ad}(a_\alpha)$ and $2e_2 \circ \mathrm{Ad}(a_\alpha)$ are non-compact. Since these two roots are strongly orthogonal, we can construct a σ-stable maximal abelian subspace $\mathfrak{j}'$ of $\mathfrak{m}$ such that

$$\dim(\mathfrak{j}' \cap \mathfrak{q}) = \dim(\mathfrak{j}_0 \cap \mathfrak{q}) + 2 = \dim(\mathfrak{j} \cap \mathfrak{q}) + 1.$$

But this contradicts that $\mathfrak{j} \cap \mathfrak{q} = \mathfrak{a}$ is maximal abelian in $\mathfrak{m} \cap \mathfrak{q}$. Thus we have proved that $(e_1 - e_2) \circ \mathrm{Ad}(a_\alpha)$ is a non-compact root of $\Sigma(\mathfrak{j}_0)$.

Put $\alpha_1 = e_1 - e_2$. Then the dense H-B double coset S_1 in $\widetilde{S}P_{\alpha_1}$ is written as $S_1 = Hx_1B$, where $\mathfrak{j}_1 = \mathrm{Ad}(x_1)\mathfrak{j}$ satisfies

$$\dim(\mathfrak{j}_1 \cap \mathfrak{q}) = \dim(\mathfrak{j}_0 \cap \mathfrak{q}) + 1 = \dim(\mathfrak{j} \cap \mathfrak{q}).$$

Take $\alpha_2, \ldots, \alpha_\ell$ so that

$$\dim_\mathbb{C} \widetilde{S}P_{\alpha_1} \cdots P_{\alpha_k} = \dim_\mathbb{C} \widetilde{S}P_{\alpha_1} \cdots P_{\alpha_{k-1}} + 1$$

for $k = 2, \ldots, \ell$. Then we have

$$\dim(\mathfrak{j}_1 \cap \mathfrak{q}) = \dim(\mathfrak{j}_2 \cap \mathfrak{q}) = \cdots = \dim(\mathfrak{j}_\ell \cap \mathfrak{q})$$

by (6.1). Hence $\beta_2, \ldots, \beta_\ell$ are complex roots and so we have

$$S'_1{}^{cl} = S'_2{}^{cl} P_{\alpha_2} = \cdots = S'_\ell{}^{cl} P_{\alpha_\ell} \cdots P_{\alpha_2} = S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_2}$$

by (6.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

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Proof of Theorem 1.7. We will show that $\widetilde{S} \subset T_j = S_j^{cl}$ for every $j \in J$. By [GM2], Lemma 2, there exists a simple root β such that $S_j \check{P}_\beta \supset S_{\mathrm{op}}$. Hence $S_j \subset S_{\mathrm{op}} P_\beta$ and therefore $S'_j \subset S'_{\mathrm{op}} P_\beta$. In particular, β is not a compact root.

If H' is of FII-type, then $\Psi(\mathfrak{j})$ is F_4-type and it consists of three compact roots and one complex root β. Hence J consists of only one element j and $S_{\mathrm{op}} P_\beta = S_{\mathrm{op}} \sqcup S_j$. So it is clear that

$$\widetilde{S} \subset T_j$$

because $\widetilde{S} \subset T_j$ for some $j \in J$ by [GM2], Theorem 2.

So we may assume that H' is not of FII-type. It is known that

$$\widetilde{S} \subset S_j^{cl} \Longleftrightarrow (\widetilde{S}')^{cl} \supset S'_j$$

([M3], Corollary, [MUV], Corollary 1.4). Suppose that $\widetilde{S} \not\subset T_j = S_j^{cl}$. Then $(\widetilde{S}')^{cl} \not\supset S'_j$. Hence

$$\alpha_1, \ldots, \alpha_\ell \neq \beta$$

by Lemma 6.1. (If H' is of CII-type, then $\widetilde{S} \not\subset T_j$ implies $S_1 \not\subset T_j$ because $\widetilde{S} \subset S_1^{cl}$. Hence $S_1'^{cl} \not\supset S'_j$ and therefore $\alpha_2, \ldots, \alpha_\ell \neq \beta$ by Lemma 6.1(ii). Since α_1 is compact, we have $\alpha_1, \ldots, \alpha_\ell \neq \beta$.) Thus we have

$$\widetilde{S} \subset S_{\mathrm{op}} P_\Theta$$

with $\Theta = \Psi - \{\beta\}$. In Remark 1.2(ii) we see that $HP_\Theta = G$ holds only when $\Theta = \Psi - \{\beta\}$ with some compact root β. So we have $HP_\Theta \neq G$ and it follows from Theorem 1.3(iii) that

$$\widetilde{S} \cap S_{\mathrm{op}} P_\Theta = \emptyset,$$

a contradiction. Thus we have proved $\widetilde{S} \subset T_j$. □

7. Appendix

In this appendix, we assume H' is a simple Lie group and G is a complexification of H'. (So G, H and H' are $G_{\mathbb{C}}$, $K_{\mathbb{C}}$ and $G_{\mathbb{R}}$, respectively, in Section 1.) In Section 7.2 we will moreover assume that H' is of non-Hermitian type.

7.1. **Lemma 5.1 in** [V]**.** First we will review a lemma due to Vogan [V] which is used frequently in this paper. This lemma is generalized for arbitrary real symmetric pairs (G, H) in [M3], Lemma 3. But we will restrict ourselves to complex symmetric pairs for simplicity.

The full flag manifold of G is identified with the set $\mathcal{F}$ of all the Borel subgroups in G. Every H-orbit in $\mathcal{F}$ contains a Borel subgroup of the form

(7.1) $$B = B(\mathfrak{j}, \Sigma^+) = Z_G(\mathfrak{j}) \exp \mathfrak{n}$$

with $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}(\mathfrak{j}, \alpha)$, where $\mathfrak{j}$ is a σ-stable maximal abelian subspace of $\mathfrak{m}$ and Σ^+ is a positive system of the root system Σ of the pair $(\mathfrak{g}, \mathfrak{j})$ ([M1], Theorem 1, [R], Theorem 13). ($\mathfrak{j}_{\mathbb{C}}$ is a σ-stable Cartan subalgebra of $\mathfrak{g}$.) By the symmetry of H and H' every H'-orbit in $\mathcal{F}$ also contains a Borel subgroup B of the form (7.1). In [M1] (Corollary of Theorem 3) we defined the natural correspondence between H-orbits S in $\mathcal{F}$ and H'-orbits S' in $\mathcal{F}$ so that S and S' contain the same Borel subgroups B of the form (7.1). (We remark here that the H'-orbit structure on the full flag manifold $\mathcal{F}$ was first explicitly studied in [A].)

Roots in Σ are usually classified as follows:
(i) If $\sigma(\alpha) = \alpha$ and $\mathfrak{g}(\mathfrak{j}, \alpha) \subset \mathfrak{h}$, then α is called a "compact root".

(ii) If $\sigma(\alpha) = \alpha$ and $\mathfrak{g}(\mathrm{j}, \alpha) \subset \mathfrak{q}$, then α is called a "non-compact root".

(iii) If $\sigma(\alpha) = -\alpha$, then α is called a "real root".

(iv) If $\sigma(\alpha) \neq \pm\alpha$, then α is called a "complex root".

For a simple root α of Σ^+, we can define a parabolic subgroup P_α by

$$P_\alpha = B \sqcup B w_\alpha B.$$

Lemma 7.1 ([V], Lemma 5.1, cf. also [M3], Lemma 3). *HP_α is decomposed into H-B double cosets as follows:*

(i) *If α is compact, then $HP_\alpha = HB$.*

(ii) *If α is non-compact, then $HP_\alpha = HB \cup Hw_\alpha B \cup Hc_\alpha B$ and $\dim_{\mathbb{C}} HB = \dim_{\mathbb{C}} Hw_\alpha B = \dim_{\mathbb{C}} Hc_\alpha B - 1$. Here $c_\alpha = \exp(X + \theta X)$ with some $X \in \mathfrak{g}(\mathrm{j}, \alpha)$ such that $c_\alpha^2 = w_\alpha$. (Sometimes HB and $Hw_\alpha B$ coincide.)*

(iii) *If α is real, then $HP_\alpha = HB \cup Hc_\alpha B \cup Hc_\alpha^{-1} B$ and $\dim_{\mathbb{C}} Hc_\alpha B = \dim_{\mathbb{C}} Hc_\alpha^{-1} B = \dim_{\mathbb{C}} HB - 1$. Here $c_\alpha = \exp(X + \theta X)$ with some $X \in \mathfrak{g}(\mathrm{j}, \alpha) \cap \mathfrak{q}'$ such that $c_\alpha^2 = w_\alpha$. (Sometimes $Hc_\alpha B$ and $Hc_\alpha^{-1} B$ coincide.)*

(iv) *If α is complex, then $HP_\alpha = HB \sqcup Hw_\alpha B$. Moreover we have*

$$\dim_{\mathbb{C}} Hw_\alpha B = \begin{cases} \dim_{\mathbb{C}} HB + 1 & \text{if } \sigma\alpha \in \Sigma^+, \\ \dim_{\mathbb{C}} HB - 1 & \text{if } \sigma\alpha \notin \Sigma^+. \end{cases}$$

7.2. **A lemma for non-Hermitian cases.** Suppose that $n = \dim_{\mathbb{C}} \mathfrak{g}(\mathfrak{t}, \alpha) = 1$ for a longest root α in $\Sigma(\mathfrak{t})$. Let $\widetilde{\mathfrak{t}}$ be a maximal abelian subalgebra of $\mathfrak{k}$ containing $\mathfrak{t}$. Then the restricted root α is the restriction of the root $\widetilde{\alpha}$ for $\widetilde{\mathfrak{t}}$ such that $\widetilde{\alpha}$ vanishes on $\mathfrak{t}_{\mathfrak{h}} = \widetilde{\mathfrak{t}} \cap \mathfrak{h}$.

As in Section 4 define a maximal abelian subspace $\mathfrak{a} = \mathbb{R}Z \oplus i\mathfrak{t}_\alpha$ of $\mathfrak{m} \cap \mathfrak{q}$. Then $\mathrm{j} = \mathfrak{a} \oplus \mathfrak{t}_{\mathfrak{h}}$ is a maximal abelian subspace of $\mathfrak{m}$. Put $\mathrm{j}' = \mathrm{Ad}(a_\alpha^{-1})\mathrm{j} = \mathbb{R}Z' \oplus i\mathfrak{t}_\alpha \oplus \mathfrak{t}_{\mathfrak{h}}$, where $Z' = \mathrm{Ad}(a_\alpha^{-1})Z$ as in Section 3. Then j' is a σ-stable maximal abelian subspace of $\mathfrak{m}$ such that $\dim(\mathrm{j}' \cap \mathfrak{q}) = \dim(\mathrm{j} \cap \mathfrak{q}) - 1$.

Lemma 7.2. *Suppose that H' is of non-Hermitian type. Then there exists a $t \in T \cap H$ such that $e^\alpha(t) = -1$ and that $\mathrm{Ad}(t)|_{\mathrm{j}'}$ is the reflection with respect to Z'.*

Proof. Clearly we may assume that G is simply connected. So the compact real form K of G is also simply connected. Then it is known that K^σ is connected. So we have

$$K^\sigma = K \cap H.$$

Note that $\Sigma(\mathfrak{t})$ is identified with the restricted root system of the compact symmetric pair $(K, K \cap H)$. Let Z_β be the element of $\mathfrak{t}$ defined by

$$Z_\beta = \frac{4\pi i\beta}{(\beta, \beta)}.$$

(This means $\gamma(Z_\beta) = 4\pi i(\gamma, \beta)/(\beta, \beta)$ for all $\gamma \in \Sigma(\mathfrak{t})$.) Then it is known that the lattice

$$\{Y \in \mathfrak{t} \mid \exp Y = e\}$$

is generated by $\{Z_\beta \mid \beta \in \Sigma(\mathfrak{t})\}$ (cf. [M6], Appendix).

Note that $\Sigma(\mathfrak{t})$ is also identified with the restricted root system of H'. It is known that H' is of Hermitian type if and only if the following two conditions are satisfied:

(i) $\Sigma(\mathfrak{t})$ is C-type or BC-type.

(ii) $\dim_{\mathbb{C}} \mathfrak{g}(\mathfrak{t}, \alpha) = 1$ for the longest roots α in $\Sigma(\mathfrak{t})$.

Since we assume H' is of non-Hermitian type and since we assume (ii), the restricted root system $\Sigma(\mathfrak{t})$ does not satisfy (i). Hence there exists a $\beta \in \Sigma(\mathfrak{t})$ such that

$$(7.2) \qquad \frac{(\alpha, \beta)}{(\beta, \beta)} = \frac{1}{2}.$$

Consider the element $t = \exp(1/2)Z_\beta$ of T. Since $t\sigma(t)^{-1} = t^2 = \exp Z_\beta = e$, we have $t \in K^\sigma = K \cap H$. By (7.2) we have

$$\alpha\left(\frac{1}{2}Z_\beta\right) = \frac{2\pi i(\alpha, \beta)}{(\beta, \beta)} = \pi i$$

and therefore $e^\alpha(t) = e^{\pi i} = -1$.

Since $Z' \in \mathfrak{m}_\mathbb{C}(\mathfrak{t}, \alpha) \oplus \mathfrak{m}_\mathbb{C}(\mathfrak{t}, -\alpha)$, we have $\mathrm{Ad}(t)Z' = -Z'$. (Here we consider the complexification of the complex Lie algebra $\mathfrak{g}$.) On the other hand $\mathrm{Ad}(t)$ acts trivially on $i\mathfrak{t}_\alpha \oplus \mathfrak{t}_\mathfrak{h}$. Hence $\mathrm{Ad}(t)$ acts on $\mathfrak{j}'$ as the reflection with respect to Z'. $\square$

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ADDED IN PROOF

Recently Conjecture 1.1 was solved in [M9] for Hermitian cases. So the conjecture is now completely solved affirmatively (cf. Remark 1.10).

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